THE UNIVERSITY OF ALBERTA

GAUGE INVARIANCE AND FOUR-FERMION INTERACTIONS

by

© ROGER P. PALMER

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF PHYSICS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled GAUGE INVARIANCE AND FOUR-FERMION INTERACTIONS, submitted by Roger F. Palmer in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

In this thesis two recurring problems of Quantum Field Theory have been investigated: The problem of a massive gauge invariant spin-one field and the problem of "ghosts" in theories that attempt to describe bound states.

The first three chapters are devoted to the gauge problem. It is shown that a skew-symmetric second rank tensor can be used to describe a spin-one quanta and that this formulation allows us to retain manifest gauge invariance. The interaction of this field with a Dirac field is considered and it is shown that gauge invariance greatly restricts the possible form of the interaction Lagrangian. It is shown that the theory obeys "Matthew's Rule" and that the mass equals zero limit exists for certain S-matrix elements. Chapter 3 is concluded with a discussion of the relationship to the usual Proca formulation of the spin-one field and its minimal coupling to the Dirac field. It is shown that our interaction is equivalent to the minimally coupled form when the gauge is fixed, except for the appearance of a four-fermion term.

In the second half of this work we have investigated some of the properties of four-fermion interactions in relation to the more general problem of bound state

"ghosts". In Chapter 4 we make a detailed analysis of a modification to the Yukawa interaction. In this model a four-fermion term is added to the usual linear coupling of the pseudoscalar meson to the Dirac field. We find that a bound state does exist but that it is not a ghost.

Finally, the propagator for the vector field interacting with a Dirac field is examined to see if a bound state exists for the theory we develop in the first half of this thesis. It is found that a bound state does not appear in this case.

Chapters 2 and 3 have been published [Takahashi and Palmer, 1970; Palmer and Takahashi, 1970] and the fourth chapter is presently being prepared for publication. Several papers have appeared in the last year that have examined some aspects of the spin-one theory we discuss. In particular, Macfarlane and Tait [1971] have done some further work on the basic formalism while Shamaly [Shamaly and Capri, 1971] and Jenkins [1972] have worked on the interaction with the electromagnetic field.

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CHAPTER 1

INTRODUCTION

Modern Physics has taken us into realms where our intuition is at best on very shaky ground: Our classical experience is often more a hindrance than a help. Under these circumstances conservation laws have taken on an increasingly central role. They express in simple concise terms a great deal of experimental data. In classical mechanics the application of the principles of conservation of energy and momentum is often the most direct method of solving a problem. These "laws" can, of course, be derived from the equations of motion and it is these that are usually considered fundamental. The change that has taken place is that in Modern Physics it is the conservation laws that are considered basic and the equations of motion, in some sense, a consequence.

One example of this change of philosophy is the case of the so called "strange" particles. Analysis of bubble-chamber photographs for high energy processes revealed that certain particles were always produced in pairs. It was postulated that, in analogy to electrical charge, these particles must possess some intrinsic charge, later called their "strangeness", that had to be conserved. The philosophy being that any event

that did not violate a conservation law <u>must</u> take place. It followed that there must be a conserved quantity that forbids the single production of these particles.

In the context of Lagrangian Field Theory, a conservation law manifests itself in the form of a symmetry property of the Lagrangian. We should not make the mistake of assuming this to be a property of Field Theory alone. Even in classical mechanics it is possible to write equations of motion that are, for example, invariant under time translation but violate the conservation of energy. We find, however, that either no Lagrangian exists for these systems or that the Lagrangian is not itself time translation invariant. In the examples we shall consider a Lagrangian always exists and the symmetry of the field equations is also present in the Lagrangian. For our purpose a conservation law implies a symmetry in the Lagrangian and vice-versa.

Whilst the conservation of the intrinsic charges such as strangeness, greatly reduces our labour in Particle Physics by creating selection rules amongst the myriad of possible processes, they cannot in themselves completely describe particle phenomena. The clearest indication of this is that these 'new' conservation laws have only a limited range of application. We know, for example, that a change in total

strangeness is possible in the weak and electromagnetic interactions. In other words the conservation laws are violated under certain circumstances. Any "final" theory of elementary particles must have this violation built-in. Little progress has been made in this direction. The strong interaction, however, conserves all these intrinsic charges exactly, so in this restricted area we may expect some degree of success if we build our theory around the conservation laws.

This approach to a theory of the strong interaction was first seriously suggested by Sakurai [1960, 1969], although the basic idea in his work can be traced to the much earlier paper by Yang and Mills [1954].

In attempting to develop a new theory we should always examine the successes of the past. The only real success in Field Theory has been Quantum Electrodynamics, so it is reasonable that one should look there for inspiration to proceed.

In Quantum Electrodynamics the conservation of electrical charge is related to the concept of gauge invariance. Not only does this simple internal symmetry of the theory lead to the conservation law but it also gives us the correct form of the interaction Lagrangian. Regarded from a more positive point of view, it could be said that the conservation

of electrical charge is the raison-d'être of the electromagnetic field and completely determines how it will interact with the charge carrying quanta. Surely we should try to find an analogous approach to the strong interaction.

In the absence of the weak and electromagnetic interactions we have three exact symmetries. Foremost there is baryon number conservation. This is an exact symmetry even in the presence of the weaker interactions and so should have very deep theoretical significance. After all, it could be said we owe our very existence to this conservation law. What else forbids the decay of the proton? Baryon number is also the only quantum number that distinguishes between the fermions that take part in the strong interactions and those that do not. The other exact symmetries that apply in the realm of the strong interaction are related to strangeness, hypercharge and isotopic spin. As strangeness and hypercharge are related through baryon number we can look on either one as a basic conserved quantity. Hypercharge is usually given this honour although there seems little a-priori reason for one choice over the other.

Isotopic spin, which was the first of these quantities to be associated with the gauge idea [Yang and

Mills, 1954], was introduced originally in connection with the charge independence of nucleon-nucleon interactions in the absence of electromagnetic effects. All experimental evidence suggests that this indeed an exactly conserved quantity for strong interactions. If we believe in Lagrangian Field Theory, the strong Lagrangian must exhibit these three symmetries.

Elementary text books state that baryon number conservation is a result of the phase invariance of the Dirac Lagrangian. The argument goes as follows:

If baryon number is conserved then there can be no unitary operator that connects a state with, say, baryon number one to a state with baryon number equal to zero. It follows that the relative phases of states with different baryon number are completely arbitrary. Physical quantities should be independent of the transformation

$$|A\rangle \rightarrow \exp\{iB\lambda\}|A\rangle \quad , \qquad (1.1)$$

where $|A\rangle$ is an eigenstate of the "baryon operator". In Lagrangian formalism this implies that the Lagrangian should be invariant under the transformation

$$\psi(x) \rightarrow e^{iB\lambda}\,\psi(x) \quad , \qquad (1.2)$$

where $\psi(x)$ is the field operator associated with the baryon carrying quanta.

Hypercharge conservation can be treated in exactly the same way, as too can isotopic spin except that in this case the associated transformation is

$$\psi(x) \rightarrow \exp\{i\vec{\tau}.\vec{\lambda}\} \quad \psi(x) \quad . \tag{1.3}$$

In the above transformations λ is assumed to be a real constant (or a constant vector in isospace) that does not depend on any particular space-time point. This is absolutely necessary for the invariance of the free Dirac Lagrangian. What does this imply? It says that if we carry out a transformation at one point with some arbitrary value of λ then a simultaneous transformation with exactly the same value of λ must take place everywhere. Yang and Mills [1954] found this very unsatisfactory. It seemed to violate all our usual ideas of locality. Why, they argued, should choice of isospin gauge (which is the equivalent of distinguishing between the proton and the neutron) at any one point mean that the choice is fixed for all space time?

To get round this difficulty Pauli introduced the concept of a gauge transformation of the second kind, now more usually called a local gauge transformation. In this λ is allowed to be a function of x. If this is done, the Dirac Lagrangian is no longer invariant unless we add an interaction term of the form

$$\mathcal{L}_{int} = -gA_\mu J_\mu(x) \quad . \tag{1.4}$$

In this equation A_μ is a vector field which transforms under the gauge transformation as

$$A_\mu(x) \rightarrow A_\mu + \partial_\mu \Lambda(x) \quad , \tag{1.5}$$

and $J_\mu(x)$ is a current that is conserved as a consequence of the invariance of the total Lagrangian.

Sakurai [1960] suggested that the strong interaction should be described by an interaction Lagrangian made-up of three terms like that in equation (1.4). There would be three vector fields and three coupling constants, one associated with each of the three basic internal charges: baryon number; hypercharge and isotopic spin. The corresponding currents would be made up of a sum of the vector current contributions from each of the charge carrying fields. For example, the baryon current, $J_\mu^B(x)$ would be,

$$J_\mu^B(x) = i\bar{\psi}_N(x)\gamma_\mu\psi_N(x) + i\bar{\psi}_\Lambda\gamma_\mu\psi_\Lambda + i\bar{\psi}_\Sigma\gamma_\mu\psi_\Sigma + \ldots. \tag{1.6}$$

There would be one additional term for each baryonic field. A key point in this argument is that the coefficient of each term would be one as each field has

the same baryon number. (If such a thing as a super-baryon with higher baryon number, n, was found in the future then it would be added in in just the same way except that its coefficient would be n.)

One can immediately appreciate the elegance of this scheme. Particles carrying the conserved charge interact through the medium of a vector field whose very presence is a consequence of the conservation law. Such a scheme exhibits so many properties that we instinctively desire in a physical theory. It has great simplicity, it is the result of a logical approach, and it contains a "dynamical manifestation" of the internal symmetry.

The idea that a symmetry should be directly related to the appearance of a field is an old and recurring theme. A model such as that due to Schwinger [1956] in which the pion field is considered as a dynamical manifestation of hypercharge conservation is a good example.

Critics of Sakurai's Theory will point out that one thing it lacks is the wide range of experimental verification of the Yukawa-type interaction. This is certainly true; but no obvious theoretical reason seems to exist for the linear coupling of the zero spin field to the baryon field. Sakurai's scheme has

the "why". What it must do in the final analysis is, of course, to reproduce all the successes of the Yukawa coupling.

On a qualitative basis the theory suggested by Sakurai had many successes and having only three parameters is certainly a tremendous improvement over most phenomological approaches. It has, however, one devastating flaw. Gauge invariance of the whole Lagrangian can only be achieved, it seems, if the vector fields have zero mass! Lee and Yang [1955] have examined the experimental consequences of the vector field associated with baryon number being massless. They conclude that such an interaction can have nothing to do with the strong coupling! Attempts have been made to overcome this difficulty: The gauge idea can be dropped and the minimally coupled equations taken as the starting point; a scalar field can be introduced to save the gauge principle using the method developed by Stueckelberg [1938] as has been suggested by Fujii [1959]. Both these approaches seem highly unsatisfactory in the light of the original idea.

Another possible solution to the zero mass problem of gauge-independent field has been suggested on many occasions since Sakurai's original paper appeared. This is to let the Heisenberg field have

zero mass to preserve the gauge invariance but have an interaction that will generate a mass term at the level of the physical fields. This approach would follow much the same philosophy as the Nambu model [Nambu and Jona-Lasino, 1961] for the fermion field. No completely successful theory of this type has been developed and any such theory requires the ad hoc introduction of self-interaction terms of some variety in the Lagrangian that would again be in contradiction to the spirit of the original idea.

More recently interest in the gauge fields has been revived by the numerous investigations of theories involving spontaneous breakdown of symmetry. The success of such theories in non-relativistic problems such as superfluidity, superconductivity and ferromagnetism, has inspired the hope that they might lead to deeper insight in the relativistic domain.

In the relativistic case we know that the spontaneous breakdown of symmetry must be accompanied by the appearance of a massless scalar particle [Nambu and Jona-Lasino, 1961; Goldstone, Salam and Weinberg, 1962]. Such particles are not seen in nature. Higgs [1964(a), 1964(b), 1966] and Kibble [1967] have suggested that this difficulty can be overcome if the conserved currents are coupled to gauge fields. This

suggestion was motivated by the observation that the B.C.S. model does not contain zero mass excitations if the Coulomb interaction is taken into account [Anderson, 1958; 1963].

The appearance of zero mass particles in these broken symmetry theories is certainly a major draw back, but it must be remembered that it is the appearance of the "Goldstone bosons" that is closely related to the preservation of the current conservation that corresponds to the original symmetry [Umezawa, 1965; Sen and Umezawa, 1967]. In a recent paper, that examined this point in detail [Aurilia and Takahashi, 1971], it was shown that even when there are gauge fields, internal consistency demands the presence of massless particles. These, in turn, become the carriers of the original symmetry transformation. A key point, that came out of this paper, is that the physical vector field does not obey the Proca equation but rather a gauge invariant massive equation. This only agrees with the Proca equation when the gauge is fixed.

A spin-one field can, of course, be described by several equations other than Proca's. In particular, by formulating the spin-one field in terms of a skew-symmetric tensor [Iwaski, 1968; Chang and Gursey, 1969; Takahashi and Palmer, 1970] we can arrive at a gauge

invariant spin-one massive equation. Such a formulation is discussed in chapter two. We will show that it is gauge invariant at least in the free field case. The equations we arrive at are of exactly the same form as those for the physical vector field that appears in Takahashi's and Aurillia's paper we discussed above.

Unfortunately, this formalism does little to progress towards a solution of Sakurai's problem as no Lagrangian has been found that will remain invariant under both a local gauge transformation of the Dirac field and the spin-one field. The Dirac Lagrangian does maintain global gauge symmetry which preserves the current conservation, while the insistence on gauge independence for the spin-one field greatly reduces the number of possible forms of the interaction Lagrangian.

While such a theory cannot be said to solve the problem of the massive gauge field, it does much that is required of any such theory. The true significance of gauge-independence seems to be that it gives a definite prescription for the selection of one form of interaction Lagrangian from the vast range possible. This formulation certainly achieves that end. It is also interesting to notice that when the gauge is fixed

in the interacting case, so as to recover the Proca equation for the spin-one field, the usual minimal coupling is also regained plus a four-fermion current-current type term. It is just such a term that could be expected to reproduce the phenomenological Yukawa interaction [Fermi and Yang, 1949].

The second half of this thesis is taken up by an investigation of four-fermion interactions. In chapter four we examine a Yukawa interaction that has been modified by the addition of a four-fermion self interaction term. The interest in this investigation is centered around the possibility that fermion-antifermion boundstates may exist.

Models in which the particle propagators have more than one pole, as is the case when we have a boundstate, usually lead to the appearance of states with negative norm. Umezawa [1956] shows that, in general, if we have 2n poles in the propagator then n will be associated with a renormalization constant that is positive but the others will be negative! This suggests that in half the cases the square of the coupling constants will be negative even if they were originally introduced in the Lagrangian as real parameters! He suggested that one way this difficulty could be overcome is if we could find a model in which

the propagator had a zero between each pole. The modified Yukawa interaction has just this structure.

Examination of the modified Yukawa interaction suggests that the propagator for the vector field should be investigated to see if it too has the structure required by Umezawa. We find, however, that in this case there is no boundstate. The principal difference between the two cases that gives rise to the different result is that in the Yukawa-type interaction we have two independent coupling constants. In the vector case, on the other hand, the coupling constants of the four fermion part and the "minimal" part are closely related.

CHAPTER 2

A GAUGE-INDEPENDENT FORMULATION OF A MASSIVE FIELD WITH SPIN ONE

In this chapter we describe a formulation for a massive spin-one field that is gauge-independent, at least in the free field case. We start with the Maxwell field equations and show that there are at least two natural generalizations which describe a massive field. One is the usual Proca equation while the other uses a skew-symmetric second rank tensor and maintains manifest gauge invariance.

A Lagrangian is constructed for the tensor field, the quantization carried out and the wave functions constructed. Finally, we show the relationship between this formulation and that of Proca.

2.1 The Field Equation

The first equations that were used to describe a spin-one field were the Maxwell equations. In the notation that is adopted throughout this thesis* they

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*The notation and conventions used in this thesis are given in Appendix A. The notation is that used by Takahashi [1969].

can be written as:

$$\partial_\mu F_{\mu\nu}(x) = 0 \tag{2.1}$$

$$\partial_\alpha F_{\mu\nu}(x) + \partial_\mu F_{\nu\alpha}(x) + \partial_\nu F_{\alpha\mu}(x) = 0 \quad . \tag{2.2}$$

Equation (2.2) implies the existence of a vector field, $A_\mu(x)$, which is defined by

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \quad . \tag{2.3}$$

Equation (2.1) can then be written as

$$(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\, A_\nu(x) = 0 \quad . \tag{2.4}$$

These equations are gauge invariant, in the sense that if we make the transformation

$$A_\mu(x) \to A_\mu^{'}(x) = A_\mu(x) + \partial_\mu\Lambda(x) \tag{2.5}$$

the new field, $A_\mu^{'}(x)$, obeys the same equations of motion as $A_\mu(x)$. This freedom can be removed by imposing the subsidiary condition

$$\partial_\mu A_\mu(x) = 0 \quad . \tag{2.6}$$

This condition has the effect of removing the spin-zero part of the vector field and leaving us with pure spin-one.

The most straightforward way to generate a field equation for a massive spin-one field is to replace (2.4) by

$$[(\Box - m^2)\delta_{\mu\nu} - \partial_\mu\partial_\nu]A_\nu(x) = 0 \quad . \tag{2.7}$$

That is to make the obvious generalization

$$\Box \to \Box - m^2 \quad . \tag{2.8}$$

Equation (2.7) is the usual field equation used for a massive spin-one field. It is referred to as the Proca equation. In making this generalization we have destroyed the gauge-invariance, but now we see that the subsidiary condition follows directly from the equation of motion. The Proca equation thus still describes pure spin-one despite the loss of the gauge freedom.

The Proca equation is not the only possible generalization of the Maxwell field. We could, for example, write equation (2.2) as

$$\Box F_{\mu\nu}(x) = 0 \quad , \tag{2.9}$$

then our generalization procedure, (2.8), leads to the equation

$$(\Box - m^2)F_{\mu\nu} = 0 \quad . \tag{2.10}$$

This in no way denies the existence of the vector field so we still have

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \ . \tag{2.11}$$

This formulation has the advantage of preserving the manifest gauge-invariance. In fact, as we will show later, the addition arbitrariness that is introduced by writing equation (2.9) can be removed by fixing the gauge. If this is done, equations (2.10)-(2.11) are equivalent to the Proca equation. Coombes [1968] has suggested that (2.10) should be used in preference to the Maxwell equations to describe the photon. He retains the mass throughout his calculations, taking the mass equals zero limit at the end. He argued that the vector potential is basically an "unphysical" object only introduced into Classical Electrodynamics to simplify the mathematics, so it is not reasonable that the vector field should become the physically important entity when we go over to quantum theory. The mass equals zero limit does not exist for the Proca formulation and it is not shown that it exists in Coombes' theory either. We will show that this limit does exist in our case for S-matrix elements involving only internal meson lines.*

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* Chapter 3, §3.2.

The possibility of describing the spin-one field by equations (2.10) and (2.11) has been known for some time [Iwasaki, 1968; Chang and Gursey, 1969] but it has proven very difficult to find a Lagrangian that will lead directly to these equations. A Lagrangian can be found, however, if we express the theory in terms of the dual tensor, $\psi_{\mu\nu}(x)$. This field obeys the equations

$$(\Box - m^2)\psi_{\mu\nu}(x) = 0 \tag{2.12}$$

$$\psi_{\mu\nu} + \psi_{\nu\mu} = 0 \tag{2.13}$$

$$\partial_\mu \psi_{\mu\nu}(x) = 0 \quad . \tag{2.14}$$

Kyriakopoulos [1969] has constructed a Lagrangian for a similar field but he had to impose the symmetry condition, (2.13), separately. Such a Lagrangian does not contain all the information about the field and thus greatly complicates the quantization procedure.* The following Lagrangian satisfies all the requirements:

$$\begin{aligned}\mathcal{L} = -\tfrac{1}{2}[&\partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\mu\nu}(x) - \partial_\lambda\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\nu\mu}(x)\\ &- \partial_\mu\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\lambda\nu}(x) + \partial_\mu\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\nu\lambda}(x)\\ &+ \partial_\nu\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\lambda\mu}(x) - \partial_\nu\bar{\psi}_{\mu\nu}(x)\partial_\lambda\psi_{\mu\lambda}(x)]\\ &- m^2\bar{\psi}_{\mu\nu}(x)\psi_{\mu\nu}(x) \quad ,\end{aligned} \tag{2.15}$$

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* See, for example, Takahashi [1969], Chapter 5.

where

$$\bar{\psi}_{\mu\nu}(x) = \psi^{\dagger}_{\lambda\tau} g_{\lambda\mu} g_{\tau\nu} \quad . \tag{2.16}$$

The Euler-Lagrange equation with respect to $\bar{\psi}_{\mu\nu}$ is

$$(\Box - m^2)[\psi_{\mu\nu}(x) - \psi_{\nu\mu}(x)] - [\partial_\mu \partial_\lambda \psi_{\lambda\nu}(x) - \partial_\nu \partial_\lambda \psi_{\lambda\mu}(x)]$$

$$+ [\partial_\mu \partial_\lambda \psi_{\nu\lambda}(x) - \partial_\nu \partial_\lambda \psi_{\mu\lambda}(x)]$$

$$- m^2[\psi_{\mu\nu}(x) + \psi_{\nu\mu}(x)] = 0 \quad . \tag{2.17}$$

It must be emphasized that the symmetry of $\bar{\psi}_{\mu\nu}$ is not assumed when taking the variation. However, if we interchange μ and ν in equation (2.17) and add the result to the original equation we find that

$$m^2[\psi_{\mu\nu}(x) + \psi_{\nu\mu}(x)] = 0 \ . \tag{2.18}$$

For $m \neq 0$ it follows that

$$\psi_{\mu\nu}(x) + \psi_{\nu\mu}(x) = 0 \ . \tag{2.19}$$

Equation (2.17) can be simplified using this result to yield

$$(\Box - m^2)\psi_{\mu\nu}(x) - \partial_\mu \partial_\lambda \psi_{\lambda\nu}(x) + \partial_\nu \partial_\lambda \psi_{\lambda\mu}(x) = 0. \tag{2.20}$$

Operating on this equation with ∂_μ, we find that

$$(\Box - m^2)\partial_\mu\psi_{\mu\nu}(x) - \partial_\lambda\Box\psi_{\lambda\nu}(x) + \partial_\lambda\partial_\nu\partial_\mu\psi_{\lambda\mu}(x) = 0. \qquad (2.21)$$

The last term in this expression is identically zero since $\psi_{\mu\nu}(x)$ is skew-symmetric. Hence, for $m \neq 0$, we get

$$\partial_\mu\psi_{\mu\nu}(x) = 0 \ . \qquad (2.22)$$

Finally if we substitute (2.21) into (2.20) we arrive at the Klein-Gordon equation for the $\psi_{\mu\nu}$ field:

$$(\Box - m^2)\psi_{\mu\nu}(x) = 0 \quad . \qquad (2.23)$$

The Euler-Lagrane equation, (2.17), contains all three equations (2.12)-(2.14) so we may express the equation of motion in the form

$$\Lambda_{\mu\nu\sigma\rho}(\partial)\psi_{\sigma\rho}(x) = 0 \ , \qquad (2.24)$$

where

$$\begin{aligned}\Lambda_{\mu\nu\sigma\rho}(\partial) &= \frac{1}{2}(\Box - m^2)(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) \\ &- \frac{1}{2}(\delta_{\nu\rho}\partial_\mu\partial_\sigma - \delta_{\nu\sigma}\partial_\mu\partial_\rho - \delta_{\mu\rho}\partial_\nu\partial_\sigma + \delta_{\mu\sigma}\partial_\nu\partial_\rho) \\ &- \frac{1}{2}m^2(\delta_{\mu\sigma}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\sigma}) \ . \qquad (2.25)\end{aligned}$$

We can now calculate the Klein-Gordon divisor, $d_{\mu\nu\sigma\rho}(\partial)$. This operator has the property

$$\Lambda_{\mu\nu\sigma\rho}(\partial)d_{\sigma\rho\alpha\beta}(\partial) \equiv (\Box - m^2)\delta_{\mu\alpha}\delta_{\nu\beta}. \tag{2.26}$$

It can readily be seen that the quantity

$$\begin{aligned} d_{\sigma\rho\lambda\tau}(\partial) &= \frac{1}{2}(\delta_{\sigma\lambda}\delta_{\rho\tau} - \delta_{\sigma\tau}\delta_{\rho\lambda}) \\ &- \frac{1}{2m^2}(\delta_{\tau\rho}\partial_\sigma\partial_\lambda - \delta_{\rho\lambda}\partial_\sigma\partial_\tau + \delta_{\sigma\lambda}\partial_\rho\partial_\tau - \delta_{\sigma\tau}\partial_\rho\partial_\lambda) \\ &- [(\Box - m^2)/2m^2](\delta_{\sigma\lambda}\delta_{\rho\tau} + \delta_{\sigma\tau}\delta_{\rho\lambda}) \end{aligned} \tag{2.27}$$

satisfies (2.26). We can, further, define the differential operator $\Gamma_{\lambda,\nu\mu\sigma\rho}(\partial,-\overleftarrow{\partial})$ by the equation

$$(\partial_\lambda + \overleftarrow{\partial}_\lambda)\Gamma_{\lambda,\mu\nu\sigma\rho}(\partial,-\overleftarrow{\partial}) \equiv \Lambda_{\mu\nu\sigma\rho}(\partial) - \Lambda_{\mu\nu\sigma o}(-\overleftarrow{\partial}). \tag{2.28}$$

That is,

$$\begin{aligned} \Gamma_{\lambda,\mu\nu\sigma\rho}(\partial,-\overleftarrow{\partial}) &= \frac{1}{2}(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma})(\partial_\lambda - \overleftarrow{\partial}_\lambda) \\ &- \frac{1}{4}[(\delta_{\nu\rho}\delta_{\lambda\sigma} - \delta_{\nu\sigma}\delta_{\lambda\rho})(\partial_\mu - \overleftarrow{\partial}_\mu) \\ &+ (\delta_{\mu\sigma}\delta_{\lambda\rho} - \delta_{\mu\rho}\delta_{\lambda\sigma})(\partial_\nu - \overleftarrow{\partial}_\nu) \\ &+ (\delta_{\nu\rho}\delta_{\lambda\mu} - \delta_{\mu\rho}\delta_{\lambda\nu})(\partial_\sigma - \overleftarrow{\partial}_\sigma) \\ &+ (\delta_{\mu\sigma}\delta_{\lambda\nu} - \delta_{\nu\sigma}\delta_{\lambda\mu})(\partial_\rho - \overleftarrow{\partial}_\rho)] . \end{aligned} \tag{2.29}$$

Hence the current

$$J_\lambda(x) \equiv \bar{\psi}_{\mu\nu}(x)\Gamma_{\lambda,\mu\nu\sigma\rho}(\partial,-\overleftarrow{\partial})\psi_{\sigma\rho}(x) \qquad (2.30)$$

is conserved, since

$$\partial_\lambda J_\lambda(x) = \bar{\psi}_{\mu\nu}(x)(\partial_\lambda + \overleftarrow{\partial}_\lambda)\Gamma_{\lambda,\mu\nu\sigma\rho}(\partial,-\overleftarrow{\partial})\psi_{\sigma\rho}(x)$$

$$= \bar{\psi}_{\mu\nu}(x)[\Lambda_{\mu\nu\sigma\rho}(\partial) - \Lambda_{\mu\nu\sigma\rho}(-\overleftarrow{\partial})]\psi_{\sigma\rho}(x)$$

$$= 0 \qquad (2.31)$$

by virtue of (2.24) and its Hermitian conjugate, i.e.,

$$\bar{\psi}_{\sigma\rho}\Lambda_{\sigma\rho\mu\nu}(-\overleftarrow{\partial}) = 0 \ . \qquad (2.32)$$

2.2 Quantization

The quantization of $\psi_{\mu\nu}(x)$ can be easily carried out using the general method described by Takahashi [1969]. Our goal is to construct the energy-momentum operator, P_μ, and the commutation relationship for the $\psi_{\mu\nu}(x)$ field so that the Heisenberg equation

$$-i\partial_\mu\psi_{\sigma\rho}(x) = [\psi_{\sigma\rho}(x), P_\mu] \qquad (2.33)$$

is consistent with the equation of motion

$$\Lambda_{\mu\nu\sigma\rho}\psi_{\sigma\rho}(x) = 0 \quad . \qquad (2.34)$$

To achieve our aim we first separate out the special dependency of the field operator. That is, we first look

for c-number solutions of (2.34). If we denote these by $U_{\underset{\sim}{p}}^{(r)}(x)$ and $V_{\underset{\sim}{p}}^{(r)}(x)$,

$$\Lambda(\partial)U_{\underset{\sim}{p}}^{(r)}(x) = 0 \tag{2.35a}$$

$$\Lambda(\partial)V_{\underset{\sim}{p}}^{(r)}(x) = 0 \tag{2.35b}$$

and $V_{\underset{\sim}{p}}^{(r)}(x)$ is the charge conjugate of $U_{\underset{\sim}{p}}^{(r)}(x)$. The label r=1,2,3 distinguishes between the different helicity states. We know* that these functions form a complete orthonormal set. The normalization conditions are

$$-i\int d\sigma_\lambda(x)\bar{U}_{\underset{\sim}{p}\mu\nu}^{(r)}(x)\Gamma_{\lambda,\mu\nu\sigma\rho}(\partial,-\overleftarrow{\partial})U_{\underset{\sim}{q}\sigma\rho}^{(s)}(x)$$

$$= \delta_{rs}\ \delta(\underset{\sim}{p}-\underset{\sim}{q}) \quad , \tag{2.36}$$

$$-i\int d\sigma_\lambda(x)\bar{V}_{\underset{\sim}{p}\mu\nu}^{(r)}(x)\Gamma_{\lambda,\mu\nu\sigma\rho}(\partial,-\overleftarrow{\partial})V_{\underset{\sim}{q}}^{(s)}(x)$$

$$= -\delta_{rs}\ \delta(\underset{\sim}{p}-\underset{\sim}{q}) \quad . \tag{2.37}$$

These normalization conditions are independent of time, as

$$\frac{\delta}{\delta\sigma(x)}\int d\sigma_\lambda \bar{U}(x)\Gamma_\lambda(\partial,-\overleftarrow{\partial})U(x) = \partial_\lambda\{\bar{U}(x)\Gamma_\lambda(\partial,-\overleftarrow{\partial})U(x)\} .$$

---

* See, for example, Takahashi [1969], Chapter 5.

$$= \bar{U}(x)(\partial_\lambda + \overleftarrow{\partial}_\lambda)\Gamma_\lambda(\partial,-\overleftarrow{\partial})U(x)$$

$$= 0 \quad . \tag{2.38}$$

The closure conditions are*

$$\sum_{r=1}^{3}\int d\underset{\sim}{p}\; U^{(r)}_{\underset{\sim}{p}\mu\nu}(x)\bar{U}^{(r)}_{\underset{\sim}{p}\sigma\rho}(x') = id_{\mu\nu\sigma\rho}(\partial)\Delta^{(+)}(x-x'), \tag{2.39}$$

$$\sum_{r=1}^{3}\int d\underset{\sim}{p}\; V^{(r)}_{\underset{\sim}{p}\mu\nu}(x)\bar{V}^{(r)}_{\underset{\sim}{p}\sigma\rho}(x') = -id_{\mu\nu\sigma\rho}(\partial)\Delta^{(-)}(x-x'). \tag{2.40}$$

Since these functions form a complete set we can expand the field operator, $\psi_{\mu\nu}(x)$, in terms of them:

$$\psi_{\mu\nu}(x) = \sum_{r=1}^{3}\int d\underset{\sim}{p}[a^{(r)}(\underset{\sim}{p})U^{(r)}_{\underset{\sim}{p}\mu\nu}(x) + b^{(r)\dagger}(\underset{\sim}{p})V^{(r)}_{\underset{\sim}{p}\mu\nu}(x)] \ . \tag{2.41}$$

We can now see that if

$$P_\mu = -\frac{1}{2}\int d\sigma_\lambda(x)\bar{\psi}_{\tau\nu}(x)(\partial_\mu-\overleftarrow{\partial}_\mu)\Gamma_{\lambda,\tau\nu\sigma\rho}(\partial,-\overleftarrow{\partial})\psi_{\sigma\rho}(x) \tag{2.42}$$

and

$$[a^{(r)}(\underset{\sim}{p}),\ a^{(r')\dagger}(\underset{\sim}{p}')] = \delta_{rr'}\ \delta(\underset{\sim}{p}-\underset{\sim}{p}') \tag{2.43}$$

$$[b^{(r)}(\underset{\sim}{p}),\ b^{(r')\dagger}(\underset{\sim}{p}')] = \delta_{rr'}\ \delta(\underset{\sim}{p}-\underset{\sim}{p}') \tag{2.44}$$

---

* The functions $\Delta^{(+)}$, $\Delta^{(-)}$ and Δ are solutions of the Klein-Gordon equation with mass m. Their properties are given in Appendix B.

then (2.33) is satisfied.

It is evident that P_μ defined by equation (2.42) is conserved, i.e.,

$$\frac{\delta P_\mu}{\delta \sigma(x)} = 0 \tag{2.45}$$

because of equation (2.28). It therefore satisfies one of the necessary conditions on the energy-momentum operator. To show that it also satisfies (2.33) we first calculate the commutation relationship for the $\psi_{\mu\nu}(x)$ field using the commutators (2.43) and (2.44).

$$\psi_{\mu\nu}(x)\bar{\psi}_{\sigma\rho}(x') - \bar{\psi}_{\sigma\rho}(x')\psi_{\mu\nu}(x)$$

$$= \sum_{rr'}\int d\underset{\sim}{p}\, d\underset{\sim}{q}\; U^{(r)}_{\underset{\sim}{p}\mu\nu}(x)\bar{U}^{(r')}_{\underset{\sim}{q}\sigma\rho}(x)\,(a^{(r)}(\underset{\sim}{p})a^{\dagger(r')}(\underset{\sim}{q})$$

$$- a^{\dagger(r')}(\underset{\sim}{q})a^{(r)}(\underset{\sim}{p})) + \sum_{rr'}\int d\underset{\sim}{p}\, d\underset{\sim}{q}\; V^{(r)}_{\underset{\sim}{p}\mu\nu}(x)\bar{V}^{(r')}_{\underset{\sim}{q}\sigma\rho}(x')$$

$$\times\; (b^{\dagger(r)}(\underset{\sim}{p})b^{(r')}(\underset{\sim}{q}) - b^{(r')}(\underset{\sim}{q})b^{\dagger(r)}(\underset{\sim}{p}))$$

$$= \sum_{r}\int d\underset{\sim}{p}\{U^{(r)}_{\underset{\sim}{p}\mu\nu}(x)\bar{U}^{(r)}_{\underset{\sim}{p}\sigma\rho}(x') - V^{(r)}_{\underset{\sim}{p}\mu\nu}(x)\bar{V}^{(r)}_{\underset{\sim}{p}\sigma\rho}(x')\}$$

$$= i d_{\mu\nu\sigma\rho}(\partial)\;\Delta(x-x') \quad . \tag{2.46}$$

The last step is obtained by combining the two closure conditions, (2.39) and (2.40).

We could now use (2.46) to prove that P_μ satisfied (2.33) directly. However, it is more informative to re-write P_μ in the form

$$P_\mu = \sum_{r=1}^{3} \int d\underset{\sim}{p}\, p_\mu [a^{(r)\dagger}(\underset{\sim}{p}) a^{r}(\underset{\sim}{p}) + b^{\dagger(r)}(\underset{\sim}{p}) b^{(r)}(\underset{\sim}{p})] \qquad (2.47)$$

where

$$p_o = (\underset{\sim}{p}^2 + m^2)^{\frac{1}{2}} \qquad (2.48)$$

which is obtained by substituting (2.41) into (2.42), using the normalization conditions (2.36) and (2.37). In this form it is easily seen that the energy is positive definite and it is obvious that P_μ does indeed satisfy (2.33).

It can be shown that if we assume the field operator forms an irreducible ring; that is that any quantity that commutes with $\psi_{\mu\nu}(x)$ at all times is a c-number; then P_μ is unique [Takahashi, 1969, Chapter 6]. The P_μ we have constructed is then the energy-momentum operator.

2.3 The Wave Functions

To complete this discussion of the free field we must construct the wave functions $U^{(r)}_{\mu\nu}(\underset{\sim}{p})$ used in the field expansion (2.41). We could, of course, use the general method outlined by Takahashi [1969] to construct them directly from the equation of motion but an easier method is available in this case.

The Klein-Gordon divisor, (2.27), can be written in the form

$$d_{\sigma\rho\lambda\tau}(\partial) = \frac{1}{2}\,[d_{\sigma\lambda}(\partial)d_{\rho\tau}(\partial) - d_{\sigma\tau}(\partial)d_{\rho\lambda}(\partial)]$$

$$- [(\Box - m^2)/2m^2][\delta_{\sigma\lambda}\delta_{\rho\tau} + \delta_{\sigma\tau}\delta_{\rho\lambda}] \qquad (2.49)$$

where $d_{\mu\nu}(\partial)$ is the Klein-Gordon divisor for the Proca field, i.e.,

$$d_{\sigma\lambda}(\partial) = \delta_{\sigma\lambda} - \frac{1}{m^2}\,\partial_\sigma\partial_\lambda \quad . \qquad (2.50)$$

This suggests that the wave functions can be constructed out of two vector wave functions.

Let the two vector wave functions be $u_\mu^{(m)}(\underset{\sim}{p})$ and $u_\nu^{(m')}(\underset{\sim}{p}')$, with m,m'=1,0,-1. The coefficients of the necessary combinations are given by the Clebsch-Gordan theorem. Hence,

$$U_{\mu\nu}^{(1)}(\underset{\sim}{p}) = [\tfrac{1}{2}\,\omega(\underset{\sim}{p})]^{\frac{1}{2}}[u_\mu^{(1)}(\underset{\sim}{p})u_\nu^{(o)}(\underset{\sim}{p}) - u_\mu^{(o)}(\underset{\sim}{p})u_\nu^{(1)}(\underset{\sim}{p})]$$

$$U_{\mu\nu}^{(o)}(\underset{\sim}{p}) = [\tfrac{1}{2}\,\omega(\underset{\sim}{p})]^{\frac{1}{2}}[u_\mu^{(1)}(\underset{\sim}{p})u_\nu^{(-1)}(\underset{\sim}{p}) - u_\mu^{(-1)}(\underset{\sim}{p})u_\nu^{(1)}(\underset{\sim}{p})]$$

$$U_{\mu\nu}^{(-1)}(\underset{\sim}{p}) = [\tfrac{1}{2}\,\omega(\underset{\sim}{p})]^{\frac{1}{2}}[u_\mu^{(-1)}(\underset{\sim}{p})u_\nu^{(o)}(\underset{\sim}{p}) - u_\mu^{(o)}(\underset{\sim}{p})u_\nu^{(-1)}(\underset{\sim}{p})]$$

(2.51)

where

$$\omega(\underset{\sim}{p}) = (\underset{\sim}{p}^2 + m^2)^{\frac{1}{2}} \quad . \qquad (2.52)$$

Using the usual closure condition for the vector wave functions,

$$\sum_{r=1,0,-1} \bar{u}_\mu^{(r)}(\underset{\sim}{p})u_\nu^{(r)}(\underset{\sim}{p}) = d_{\mu\nu}(ip)/2\omega(\underset{\sim}{p}) \ , \tag{2.53}$$

we obtain

$$\sum_{r=1,0,-1} U_{\mu\nu}^{(r)}(\underset{\sim}{p})\bar{U}_{\sigma\rho}^{(r)}(\underset{\sim}{p})$$

$$= \frac{1}{2}[d_{\mu\sigma}(ip)d_{\nu\rho}(ip) - d_{\mu\rho}(ip)d_{\nu\sigma}(ip)]/2\omega(\underset{\sim}{p})$$

$$= d_{\mu\nu\sigma\rho}(ip)/2\omega(\underset{\sim}{p}) \ . \tag{2.54}$$

This agrees with the Fourier transform of (2.48) on the mass shell.

2.4 The Relationship to the Proca Formalism

In section one of this chapter we have shown that a massive field with spin one can be formulated in terms of a second rank skew symmetric tensor. A Lagrangian has been constructed and the quantization carried out. The subsidiary conditions (2.13)-(2.14) imply the existence of a vector, $V_\mu(x)$, such that

$$\psi_{\mu\nu}(x) = \frac{i}{2}\,\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma V_\rho \ . \tag{2.55}$$

The theory is, therefore, invariant under the gauge transformation

$$V_\mu \rightarrow V_\mu'(x) = V_\mu(x) + \partial_\mu\Lambda(x) \tag{2.56}$$

for any scalar function $\Lambda(x)$. We will now show that the field, $V_\mu(x)$, agrees with the Proca field when this gauge freedom is removed.

Let us introduce two three-vectors $\underset{\sim}{E}$ and $\underset{\sim}{B}$ defined by the equations

$$E_i(x) = \varepsilon_{ijk}\psi_{jk}(x) \tag{2.57}$$

$$B_j(x) = -i\ \psi_{j4}(x) \tag{2.58}$$

where $i,j,k = 1,2,3$ and ε_{ijk} is the totally skew-symmetric tensor in three dimensions. Then equation (2.14) becomes

$$\underset{\sim}{\nabla}.\underset{\sim}{B}(x) = 0 \tag{2.59}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E}(x) = -\frac{\partial \underset{\sim}{B}(x)}{\partial t} \quad . \tag{2.60}$$

The vector field is related to $\underset{\sim}{E}$ and $\underset{\sim}{B}$ by

$$\underset{\sim}{B}(x) = \underset{\sim}{\nabla} \times \underset{\sim}{V}(x) \tag{2.61}$$

and

$$\underset{\sim}{E}(x) = -\frac{\partial \underset{\sim}{V}(x)}{\partial t} - \underset{\sim}{\nabla} V_o(x) \quad . \tag{2.62}$$

On the other hand, the Klein-Gordon equation for $\psi_{\mu\nu}(x)$ can be written as

$$(\Box - m^2)\ \underset{\sim}{B}(x) = 0 \tag{2.63}$$

$$(\Box - m^2)\ \underset{\sim}{E}(x) = 0 \quad . \tag{2.64}$$

From (2.63) we have

$$
\begin{aligned}
0 &= m^2\underset{\sim}{B} - \nabla^2\underset{\sim}{B} + \frac{\partial^2\underset{\sim}{B}}{\partial t^2} \\
&= m^2\underset{\sim}{B} + \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}) - \underset{\sim}{\nabla}(\underset{\sim}{\nabla}.\underset{\sim}{B}) + \frac{\partial^2\underset{\sim}{B}}{\partial t^2} \\
&= m^2\underset{\sim}{B} + \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{B}) - \underset{\sim}{\nabla} \times \frac{\partial\underset{\sim}{E}}{\partial t} \\
&= \nabla \times [m^2\underset{\sim}{V} + \underset{\sim}{\nabla} \times \underset{\sim}{B} - \frac{\partial\underset{\sim}{E}}{\partial t}] \quad .
\end{aligned}
\tag{2.65}
$$

Hence

$$
\underset{\sim}{\nabla} \times \underset{\sim}{B} - \frac{\partial\underset{\sim}{E}}{\partial t} + m^2\underset{\sim}{V} = m^2\underset{\sim}{\nabla}\chi \quad . \tag{2.66}
$$

Similarly, from (2.64) we obtain

$$
\underset{\sim}{\nabla}.\underset{\sim}{E} + m^2 V_o = - m^2 \frac{\partial\chi}{\partial t} \quad , \tag{2.67}
$$

where $\chi(x)$ is an arbitrary scalar function.

Equations (2.66) and (2.67) are just the Proca equations except for the arbitrary gauge χ. Thus, if we fix the gauge we regain the Proca Equations.

There is another way of showing the equivalence of our formulation to that of Proca. This method illustrates the relationship between our theory and the more usual tensor formulation of the spin-one field in a more direct manner than the one we have given already: We will show that both sets of equations are

derivable from the same primitive version [Macfarlane and Tait, 1971].

Let us consider the Bargmann-Wigner equations for spin-one:

$$(\gamma_\mu \partial_\mu + m)\psi(x) = 0 \; ; \tag{2.68}$$

$$\psi(x)(\gamma^t_\mu \overleftarrow{\partial}_\mu + m) = 0 \; . \tag{2.69}$$

The γ_μ's that appear in these equations are the usual Dirac matrices and γ^t_μ is the transpose of γ_μ. The field operator, $\psi(x)$, is the totally symmetric spinor of second rank.

As the field is a 4×4 symmetric matrix in spinor space we can expand it in terms of the sixteen independent combinations of the Dirac matrices. Care must be taken to preserve the symmetry. In fact the symmetry condition implies that there are only two possible expansions:

$$\psi(x) = im\gamma_\mu CA_\mu(x) + \frac{1}{2}\,\sigma_{\lambda\nu}CF_{\lambda\nu}(x) \tag{2.70}$$

and

$$\psi(x) = im\gamma_\mu CA_\mu(x) + \frac{1}{2}\,\gamma_5\sigma_{\lambda\nu}CG_{\lambda\nu} \quad . \tag{2.71}$$

In these expansions A_μ is a four vector, $F_{\mu\nu}$ and $G_{\mu\nu}$ are skew-symmetric tensors, and C is the charge conjugation matrix which has been introduced because

$(\gamma_\mu C)^t = \gamma_\mu C$, etc. thus preserving the symmetry.

If we take the expansion (2.70) and substitute it into (2.68) and (2.69) we get

$$0 = im[\gamma_\nu,\gamma_\lambda]C\partial_\nu A_\lambda + \frac{1}{2}[\gamma_\mu,\sigma_{\lambda\nu}]C\partial_\mu F_{\lambda\nu}$$

$$+ 2im^2\gamma_\lambda CA_\lambda + m\sigma_{\lambda\nu}CF_{\lambda\nu}$$

$$= -2m\sigma_{\lambda\nu}C\partial_\lambda A_\nu + 2i\gamma_\lambda C\partial_\nu F_{\lambda\nu}$$

$$+ 2im^2\gamma_\lambda CA_\lambda + m\sigma_{\lambda\nu}CF_{\lambda\nu} \qquad (2.72)$$

where we have made use of the identity

$$[\gamma_\mu,\sigma_{\lambda\nu}] = 2i\delta_{\mu\nu}\gamma_\lambda - 2i\delta_{\mu\lambda}\gamma_\nu \quad . \qquad (2.73)$$

Setting the coefficients of γ_λ and $\sigma_{\lambda\nu}$ equal to zero we get the coupled equations

$$F_{\lambda\nu} = \partial_\lambda A_\nu - \partial_\nu A_\lambda \quad , \qquad (2.74)$$

$$\partial_\lambda F_{\lambda\nu} = \mu^2 A_\nu \quad . \qquad (2.75)$$

The Proca equation is obtained by eliminating $F_{\mu\nu}$ from (2.75) using (2.74), hence

$$\Box A_\nu - \partial_\nu\partial_\lambda A_\lambda = \mu^2 A_\nu \quad . \qquad (2.76)$$

Alternatively we could eliminate A_μ and obtain

$$m^2 F_{\lambda\nu} = \partial_\lambda\partial_\mu F_{\mu\nu} - \partial_\nu\partial_\mu F_{\mu\lambda} \quad , \qquad (2.77)$$

or,

$$(\Box - m^2)F_{\mu\nu} = 0 \ , \tag{2.78}$$

$$\varepsilon_{\alpha\beta\mu\nu}\partial_\beta F_{\mu\nu} = 0 \ , \tag{2.79}$$

$$F_{\mu\nu} + F_{\nu\mu} = 0 \ . \tag{2.80}$$

The system of equations (2.78)-(2.80) is the usual tensor formulation of the spin-one field.

If, on the other hand, we use the expansion (2.71) we get

$$\varepsilon_{\alpha\beta\sigma\rho}\partial_\beta G_{\sigma\rho} = m^2 A_\alpha \tag{2.81}$$

$$G_{\alpha\beta} = -\frac{1}{2}\ \varepsilon_{\alpha\beta\sigma\rho}\partial_\sigma A_\rho \tag{2.82}$$

which imply

$$(\Box - m^2)G_{\alpha\beta} + \partial_\alpha\partial_\sigma G_{\beta\sigma} - \partial_\sigma\partial_\beta G_{\alpha\sigma} = 0 \tag{2.83}$$

or equivalently

$$(\Box - m^2)G_{\mu\nu} = 0 \tag{2.84}$$

$$\partial_\mu G_{\mu\nu} = 0 \tag{2.85}$$

$$G_{\mu\nu} + G_{\nu\mu} = 0 \ . \tag{2.86}$$

But these equations are just those we have derived from the Lagrangian (2.15).

Since we have established the existence of the vector field $V_\mu(x)$, it is interesting to see how the various relationships derived in section 2 can be expressed in terms of this field.

For example, the commutator

$$[\psi_{\mu\nu}(x), \bar{\psi}_{\sigma\rho}(x')] = id_{\mu\nu\sigma\rho}(\partial)\Delta(x-x')$$

becomes

$$[V_\rho(x), \bar{V}_\sigma(x')] = \frac{2i}{m^2}\,\delta_{\sigma\rho}\Delta(x-x') + \partial_\rho\partial_\sigma f(x-x') \ .$$

To compare this result with the Proca commutator, it must be noted that V_ρ defined by (2.55) does not have the same dimensions as the Proca field. If this is taken into account then (2.87) is the same as in the Proca case except for the gauge freedom expressed in the arbitrary function $f(x-x')$ in place of $\Delta(x-x')$.

We have also found the energy-momentum operator, P_μ. This becomes

$$P_\mu = -\frac{1}{8}\int d\sigma_\lambda [\bar{W}_{\beta\lambda}(\partial_\mu - \overleftarrow{\partial}_\mu)(\partial_\alpha - \overleftarrow{\partial}_\alpha)W_{\alpha\beta}$$

$$+ \bar{W}_{\alpha\beta}(\partial_\mu - \overleftarrow{\partial}_\mu)(\partial_\alpha - \overleftarrow{\partial}_\alpha)W_{\beta\lambda}] \ . \tag{2.88}$$

where

$$W_{\alpha\beta} = (\partial_\alpha V_\beta - \partial_\beta V_\alpha) \ . \tag{2.89}$$

Again, this expression indicates the manifest gauge invariance that we have maintained throughout.

This concludes our discussion of the free field equations. We will now turn our attention to the interaction of the tensor spin-one field with a Dirac field.

CHAPTER 3

THE INTERACTION OF THE GAUGE-INDEPENDENT FIELD WITH A DIRAC FIELD

The free field equations for a massive field with spin one that preserve manifest gauge invariance have been discussed in Chapter 2. We now investigate the interaction of this field with a Dirac field, $B(x)$.

In this theory we are unable to maintain the local gauge invariance of the Dirac field although we do, of course, retain the global gauge invariance that ensures the conservation of the current

$$j_\mu = -i\bar{B}(x)\gamma_\mu B(x) \quad . \tag{3.1}$$

While we lose the connection between gauge invariance and the form of the interaction Lagrangian that is the rewarding feature of Quantum Electrodynamics we find that maintenance of our gauge principle does impose very strong restrictions on the form of the interaction. Further, if we fix the gauge in this theory so as to regain the Proca equations for the spin-one field we find that the fermion is coupled "minimally" to the vector field.

Another interesting feature of this formulation is that the mass equals zero limit does exist for S-matrix elements that contain only internal meson lines.

3.1 The Interaction Lagrangian

The statement that our formulation is gauge-independent depends critically on the existence of a vector field. That is, for the gauge-invariance to have any significance we must be able to make the identification

$$\psi_{\mu\nu}(x) = \frac{i}{2}\,\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma V_\rho(x) \tag{3.2}$$

that we have discussed in the previous chapter.

In the free field case we have shown that this identification is always possible, but when an interaction is present the situation is not so clear.

In general the propagator for the $\psi_{\mu\nu}$ field is

$$<0|T^*(\psi_{\mu\nu}(x),\bar{\psi}_{\sigma\rho}(x')|0> = id_{\mu\nu\sigma\rho}(\partial)\Delta_c(x-x') \;. \tag{3.3}$$

We can see that as the Klein-Gordon divisor can be expressed in the form (2.49) we may expect to have both spin-zero and spin-two propagating as well as the desired spin-one. If this is the case we will have no vector field in our theory.

Let us assume that the equations of motion become

$$\Lambda_{\mu\nu\sigma\rho}(\partial)\psi_{\sigma\rho}(x) = J_{\mu\nu}(x) \tag{3.4}$$

when the interaction is switched on. Then the existence of the vector field depends on our ability to derive the subsidiary conditions (2.13) and (2.14) from (3.4). This will only be possible if

$$J_{\mu\nu} + J_{\nu\mu} = 0 \tag{3.5}$$

and

$$\partial_\mu J_{\mu\nu}(x) = 0 \; . \tag{3.6}$$

This means that there exist vector or pseudovector currents or their linear combinations J_ρ, such that

$$J_{\mu\nu}(x) = \frac{i}{2}\, \varepsilon_{\mu\nu\sigma\rho}\partial_\sigma J_\rho \quad , \tag{3.7}$$

where J_ρ is not necessarily conserved itself.

A Lagrangian can be found that leads to an equation of motion with a source term that satisfies (3.5) and (3.6). For simplicity we shall consider the real field case. Then the required Lagrangian is

$$\mathcal{L} = \mathcal{L}_o + \mathcal{L}_D + \mathcal{L}_{int} \quad , \tag{3.8}$$

where $\mathcal{L}_o$ is the free-field Lagrangian for the $\psi_{\mu\nu}(x)$ field given by (2.15), $\mathcal{L}_D$ is the free Dirac Lagrangian and $\mathcal{L}_{int}$ is*

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* See Appendix A §14.

$$\mathcal{L}_{int} = G\varepsilon_{\mu\nu\sigma\rho}[\partial_\sigma \underset{\sim}{\psi}_{\mu\nu}(x)][\underset{\sim}{\bar{B}}(x)\gamma_\rho \underset{\sim}{B}(x)] \quad . \tag{3.9}$$

This Lagrangian leads to the equation of motion

$$\Lambda_{\mu\nu\sigma\rho}(\partial)\underset{\sim}{\psi}_{\sigma\rho}(x) = \underset{\sim}{J}_{\mu\nu}(x) \tag{3.10}$$

with

$$\begin{aligned} \underset{\sim}{J}_{\mu\nu}(x) &= G\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma[\underset{\sim}{\bar{B}}(x)\gamma_\rho \underset{\sim}{B}(x)] \\ &= G\partial_\sigma[\varepsilon_{\mu\nu\sigma\rho}\underset{\sim}{\bar{B}}(x)\gamma_\rho \underset{\sim}{B}(x)] \ , \end{aligned} \tag{3.11}$$

or

$$\underset{\sim}{J}_{\mu\nu}(x) \equiv G\partial_\sigma \underset{\sim}{J}_{\mu\nu\sigma}(x) \quad . \tag{3.12}$$

The source, $J_{\mu\nu}(x)$, satisfies (3.5) and (3.6) and also

$$\underset{\sim}{\bar{J}}_{\mu\nu}(x) = \underset{\sim}{J}_{\mu\nu}(x) \quad . \tag{3.13}$$

The fermion (Dirac) field $\underset{\sim}{B}(x)$ obeys the Dirac equation

$$-(\gamma_\mu\partial_\mu + M)\underset{\sim}{B}(x) = \underset{\sim}{J}(x) \tag{3.14}$$

where

$$\underset{\sim}{J}(x) = -G\varepsilon_{\mu\nu\sigma\rho}(\partial_\sigma \underset{\sim}{\psi}_{\mu\nu}(x))\gamma_\rho \underset{\sim}{B}(x) \ . \tag{3.15}$$

From these equations it follows that the fermion current

$$J_{\underset{\sim}{\mu}}(x) = -i\,\underset{\sim}{\bar{B}}(x)\gamma_{\mu}\underset{\sim}{B}(x) \tag{3.16}$$

is conserved

$$\partial_{\mu}J_{\mu}(x) = 0 \quad . \tag{3.17}$$

3.2 Determination of the Interaction Hamiltonian

To determine the interaction Hamiltonian we use the method originally due to Källén [1950] and Yang and Feldman [1950]*.

In this programme we must first define two auxiliary fields $\psi_{\mu\nu}(x,\sigma)$ and $B(x,\sigma)$ in terms of the Heisenberg operators and then express the sources $\underset{\sim}{J}_{\mu\nu}(x)$ and $\underset{\sim}{J}(x)$ in terms of these fields.

The auxiliary fields are defined by

$$\psi_{\mu\nu}(x,\sigma) = \psi_{\mu\nu}^{(o)}(x) + G\int_{-\infty}^{\sigma} d^4x'\,\partial_{\sigma}d_{\mu\nu\alpha\beta}(\partial)\Delta^{ret}(x-x')\underset{\sim}{J}_{\alpha\beta\sigma}(x') \tag{3.18}$$

and

$$B(x,\sigma) = B^{(o)}(x) - \int_{-\infty}^{\sigma} d^4x'\; S^{ret}(x-x')\underset{\sim}{J}(x') \quad . \tag{3.19}$$

In these equations $B^{(o)}(x)$ and $\psi_{\mu\nu}^{(o)}(x)$ are related to the asymptotic solutions of the equations (3.14) and (3.10). The space-like surface, σ, may or may not pass

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* See, also, Takahashi and Umezawa, 1953.

through the point x. If x does lie on σ we can establish that

$$\psi_{\mu\nu}(x/\sigma) = \psi^{o}_{\mu\nu}(x) + G\int_{-\infty}^{\infty} d^4x' \; \theta(x_o - x'_o)\partial_\sigma \times$$

$$d_{\mu\nu\alpha\beta}(\partial)\Delta(x-x')\underset{\sim}{J}_{\alpha\beta\sigma}(x') \tag{3.20}$$

and

$$B(x/\sigma) = B^{o}(x) - \int_{-\infty}^{\infty} \theta(x_o - x'_o)S(x-x')\underset{\sim}{J}(x') \; . \tag{3.21}$$

(We have used the notation $\psi_{\mu\nu}(x/\sigma)$ to indicate that now x lies on σ.) On the other hand, we can write down the formal solutions of the equations of motion as

$$\underset{\sim}{\psi}_{\mu\nu}(x) = \psi^{(o)}_{\mu\nu}(x) + G\int_{-\infty}^{\infty} d^4x' \partial_\sigma d_{\mu\nu\alpha\beta}(\partial)\Delta^{ret}(x-x')\underset{\sim}{J}_{\alpha\beta\sigma}(x') \tag{3.22}$$

and

$$\underset{\sim}{B}(x) = B^{(o)}(x) - \int_{-\infty}^{\infty} d^4x' \; S^{ret}(x-x')\underset{\sim}{J}(x') \; . \tag{3.23}$$

In writing these solutions we have in fact assumed that $B^{(o)}(x)$ and $\psi^{(o)}_{\mu\nu}(x)$ are proportional to the asymptotic solutions. This point will be examined again later but in the final analysis the justification can only be *a posteriori*.

We may now eliminate the $B^{(o)}$ and $\psi^{(o)}_{\mu\nu}(x)$ fields to express the Heisenberg fields in terms of the auxiliary fields. It follows that

$$\psi_{\mu\nu}(x/\sigma) = \underset{\sim}{\psi}_{\mu\nu}(x) + G\int_{-\infty}^{\infty} d^4x'[\theta(x_o-x_o'),\partial_\alpha d_{\mu\nu\sigma\rho}(\partial)] \times \Delta(x-x')\underset{\sim}{J}_{\sigma\rho\alpha}(x') \quad . \tag{3.24}$$

Making use of the explicit form of the Klein-Gordon divisor we find that the second term on the right hand side of equation (3.24) vanishes leaving

$$\psi_{\mu\nu}(x/\sigma) = \underset{\sim}{\psi}_{\mu\nu}(x) \quad . \tag{3.25}$$

In exactly the same way we find that

$$B(x/\sigma) = \underset{\sim}{B}(x) \quad . \tag{3.26}$$

However,

$$\partial_\tau\psi_{\mu\nu}(x/\sigma) = \partial_\tau\underset{\sim}{\psi}_{\mu\nu}(x) + G\int_{-\infty}^{\infty} d^4x'[\theta(x_o-x_o'),\partial_\tau\partial_\alpha d_{\mu\nu\sigma\rho}(\partial)]\Delta(x-x')\underset{\sim}{J}_{\sigma\rho\alpha}(x') \quad . \tag{3.27}$$

In this case the second term does not vanish. A lengthy calculation yields the result

$$\partial_\tau\psi_{\mu\nu}(x/\sigma) = \partial_\tau\underset{\sim}{\psi}_{\mu\nu}(x) + Gn_\tau n_\alpha\underset{\sim}{J}_{\mu\nu\alpha}(x) \tag{3.28}$$

where

$$n_\mu n_\mu = -1 \quad . \tag{3.29}$$

Here we are left with explicit normal dependent terms.

With the aid of equations (3.25), (3.26) and (3.28) we can express the sources in terms of the auxiliary fields:

$$\begin{aligned} J(x/\sigma) = & -G\,\varepsilon_{\mu\nu\tau\rho}(\partial_\tau \psi_{\mu\nu}(x/\sigma))\gamma_\rho B(x/\sigma) \\ & +G^2 \varepsilon_{\mu\nu\tau\rho}\varepsilon_{\mu\nu\alpha\beta} n_\tau n_\alpha \bar{B}(x/\sigma)\gamma_\sigma B(x/\sigma)\gamma_\rho B(x/\sigma) ; \end{aligned} \tag{3.30}$$

$$J_{\mu\nu}(x/\sigma) = G\partial_\sigma \varepsilon_{\mu\nu\sigma\rho}\bar{B}(x/\sigma)\gamma_\rho B(x/\sigma) \quad . \tag{3.31}$$

It is assumed that the auxiliary fields obey the commutation relationships.

$$[\psi_{\mu\nu}(x/\sigma),\psi_{\sigma\rho}(x'/\sigma)] = id_{\mu\nu\sigma\rho}(\partial)\Delta(x-x') \tag{3.32}$$

$$\{B_\alpha(x/\sigma),\ \bar{B}_\beta(x'/\sigma)\} = id_{\alpha\beta}(\partial)\ \Delta(x-x') \tag{3.33}$$

$$\{B_\alpha(x/\sigma),\ B_\beta(x'/\sigma)\} = 0 \quad . \tag{3.34}$$

As we can define a similar set of operators for each space-like surface σ, and we assume that these too obey the commutators (3.32)-(3.34), there must exist a unitary operator, $U(\sigma,\sigma')$, such that

$$B(x/\sigma) = U^\dagger(\sigma,\sigma')B(x/\sigma')U(\sigma,\sigma') \ , \tag{3.35}$$

so long as σ and σ' are finite. In the limit $\sigma \to -\infty$ we can only rely on experience [Ezawa, 1963] which suggests*

$$\operatorname*{Lt}_{\sigma\to-\infty} B(x/\sigma) = B^{O}(x) = z^{\frac{1}{2}} B_{in}(x) \tag{3.36}$$

where

$$\Lambda(\partial)\; B_{in}(x) = 0 \tag{3.37}$$

$$\{B_{in}(x),\bar{B}_{in}(x')\} = id(\partial)\Delta(x-x') \tag{3.38}$$

$$\{B_{in}(x),B_{in}(x')\} = 0\ . \tag{3.39}$$

The constant z is a c-number to be determined by the self consistency of the theory. (Equation (3.36) is the justification for writing the formal solutions of the equations of motion in the forms (3.22) and (3.23).)

Thus, we may write

$$B(x/\sigma) = U^{\dagger}(\sigma)B_{in}(x)U(\sigma) \tag{3.40}$$

where

$$U(\sigma) \equiv U(\sigma,\ -\infty) \tag{3.41}$$

$$U^{\dagger}(\sigma) \equiv U^{\dagger}(\sigma,\ -\infty)\ . \tag{3.42}$$

---

* Limits involving field quantities such as that in equation (3.36) are always to be taken in the "weak" sense, i.e. as limits of the matrix elements between wave packets, rather than as limits on the operator itself.

It is reasonable to assume that

$$U(-\infty) = 1 \tag{3.43}$$

$$U^{\dagger}(-\infty) = 1 \tag{3.44}$$

and that

$$U^{\dagger}(\sigma)U(\sigma) = 1 \tag{3.45}$$

$$U(\sigma)U^{\dagger}(\sigma) = 1 \ . \tag{3.46}$$

In general we can see that for an arbitrary functional of the field quantities $F(x/\sigma)$ we have

$$F(x/\sigma) = U^{\dagger}(\sigma)F(x)U(\sigma) \ . \tag{3.47}$$

The relationship (3.45) implies

$$i\,\frac{\delta U(\sigma)}{\delta\sigma}\,U^{\dagger}(\sigma) = -i\,U(\sigma)\,\frac{\delta U^{\dagger}(\sigma)}{\delta\sigma} \tag{3.48}$$

so we may put

$$i\,\frac{\delta U(\sigma)}{\delta\sigma} = \mathcal{H}(n,x)\,U(\sigma) \tag{3.49}$$

where $\mathcal{H}(n,x)$ is a functional of n_μ and $B_{in}(x)$ and its derivatives. To be consistent with (3.48) it must be Hermitian and

$$U^{\dagger}(\sigma)\mathcal{H}(n,x)U(\sigma) = \mathcal{H}(n,x/\sigma) \ . \tag{3.50}$$

Equation (3.49) implies the integrability condition

$$[i \frac{\partial}{\partial\sigma(x)} - \mathcal{A}(n,x), i \frac{\partial}{\partial\sigma(x')} - \mathcal{A}(n,x')] = 0 \quad (3.51)$$

for

$$(x-x')^2 \geq 0 \quad .$$

Here the symbol $\partial/\partial\sigma(x)$ stands for derivative of the explicit $n_\mu(x)$'s in $\mathcal{A}(n,x)$.

We can now determine $\mathcal{A}(n,x)$ as

$$i \frac{\delta B(x,\sigma)}{\delta\sigma(x')} = U^\dagger(\sigma)[B_{in}(x), \mathcal{A}(n,x')]U(\sigma) \quad (3.52)$$

and because of (3.47)

$$i \frac{\delta B(x,\sigma)}{\delta\sigma(x')} = [B(x/\sigma), \mathcal{A}(n,x'/\sigma)] \quad . \quad (3.53)$$

Finally we can use the definition of the auxiliary field to get

$$[B(x,\sigma), \mathcal{A}(n,x'/\sigma)] = id(\partial)\Delta(x-x')\underset{\sim}{J}(x') \quad . \quad (3.54)$$

The same reasoning can, of course, be followed through for $\psi_{\mu\nu}(x/\sigma)$. Hence,

$$[\psi_{\mu\nu}(x/\sigma), \mathcal{A}(n,x'/\sigma)] = id_{\mu\nu\sigma\rho}(\partial)\Delta(x-x')\underset{\sim}{J}_{\sigma\rho}(x') \quad . \quad (3.55)$$

The right hand sides of equations (3.54) and (3.55) can be expressed in terms of the auxiliary fields using the relationships (3.30) and (3.31). Once this is done

we can determine $\mathcal{H}(n,x/\sigma)$ using the commutation relationships for the auxiliary fields we have assumed. Then $\mathcal{H}(n,x)$ is found merely by replacing $B(x/\sigma)$, $\psi_{\mu\nu}(x/\sigma)$ by the asymptotic fields $B_{in}(x)$ and $\psi_{\mu\nu in}(x)$. The general theory shows* that this is the required interaction Hamiltonian.

Following this scheme and making the substitution

$$J_\rho = -i\bar{B}_{in}(x)\gamma_\rho B_{in}(x) \tag{3.56}$$

we find after a very lengthy calculation that

$$\mathcal{H}_{int}(x,n) = -\mathcal{L}_{int}(x) + G^2[J_\rho J_\rho + (n.J)^2] \ . \tag{3.57}$$

We have also performed the calculation including explicit isospin index reaching an exactly analogous result.

3.3 <u>The S-Matrix</u>

Now that we have calculated the interaction Hamiltonian we can write down the S-matrix. To second order in G the S-matrix is

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* Takahashi [1969], Chapter 8, pp.207.

$$S = 1 + i \int d^4x' \; : \mathcal{L}_{int}(x') :$$

$$- i \int d^4x' \left(\frac{1}{2} G^2\right) n_\tau n_\alpha J_{\mu\nu\alpha}(x') J_{\mu\nu\tau}(x')$$

$$+ \frac{(-i)^2}{2!} \int d^4x' \int d^4x'' \; T(\mathcal{L}_{int}(x), \mathcal{L}_{int}(x'')) \; . \qquad (3.58)$$

The fourth term in this expression yields

$$\frac{(-i)^2}{2!} \int d^4x' \int d^4x'' \; T^*(\mathcal{L}_{int}(x'), \mathcal{L}_{int}(x''))$$

$$+ \frac{(-i)^2}{2!} \int d^4x' \int d^4x'' iG^2 \varepsilon_{\mu\nu\sigma\rho} \varepsilon_{\mu\nu\sigma'\rho'} n_{\sigma'} n_\sigma$$

$$\times \; \delta^4(x'-x'') T^*(J_\rho(x'), \; J_{\rho'}(x''))$$

$$= \frac{(-i)^2}{2!} \int d^4x' \int d^4x'' T^*(\mathcal{L}_{int}(x'), \mathcal{L}_{int}(x''))$$

$$+ i \int d^4x' \left(\frac{1}{2} G^2\right) n_\sigma n_\alpha J_{\mu\nu\sigma}(x') J_{\mu\nu\alpha}(x'') \quad . \qquad (3.59)$$

So,

$$S = 1 + \int d^4x' \; : \mathcal{L}_{int}(x') :$$

$$+ \frac{(-i)^2}{2!} \int d^4x' \int d^4x'' \; T^*(\mathcal{L}_{int}(x'), \mathcal{L}_{int}(x''))$$

$$+ \; \ldots\ldots \qquad (3.60)$$

We can see that in a S-matrix calculation, at least to second order, we can replace $\mathcal{H}_{int}$ by

$$(\mathcal{H}_{int})_{effective} = -\mathcal{L}_{int} \tag{3.61}$$

so long as we also replace the T-product by the T* product.† This property of the S-matrix seems to be generally true and has been referred to as "Matthew's Rule".

In the paper by Kyriakopoulos [1969] where a similar spin-one theory was discussed the interaction Hamiltonian does not reduce to $-\mathcal{L}_{int}$ in S-matrix calculations. This result seems to stem from the fact that he had to impose the symmetry condition (2.13) separately so his Lagrangian did not contain the complete information about the field. In addition he made no attempt to preserve the gauge-invariance so he also considered an interaction term $\bar{B}\sigma_{\mu\nu}T_{\mu\nu}B$, where $T_{\mu\nu}$ is the skew-symmetric tensor field. This type of term is excluded from our theory by the conditions (3.5) and (3.6).

We have mentioned that the mass equals zero limit does not usually exist for spin-one theories. This is because of the appearance of $(m^2)^{-1}$ terms in the S-matrix elements. In this theory however it is interesting to note that the mass equals zero limit does exist for internal meson lines.

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† The T-product and T^*-product are defined in Appendix A.

To illustrate this, let us consider fermion scattering with one-meson exchange. To second order in G the S-matrix is

$$S^{(2)} = G^2\iint d^4x' \, d^4x'' \partial'_\sigma J_\rho(x')$$

$$\langle T^*(\varepsilon_{\sigma\rho\mu\nu}\bar{\psi}_{\mu\nu}(x'),\varepsilon_{\lambda\tau\xi\eta}\psi_{\lambda\tau}(x'')\rangle_o \partial''_\xi J_\eta(x''). \qquad (3.62)$$

The contraction term is

$$\langle 0|T^*(\varepsilon_{\sigma\rho\mu\nu}\varepsilon_{\lambda\tau\xi\eta}\bar{\psi}_{\mu\nu}(x'),\ \psi_{\lambda\tau}(x''))|0\rangle$$

$$= \varepsilon_{\sigma\rho\mu\nu}\varepsilon_{\lambda\tau\xi\eta}\langle 0|T^*(\bar{\psi}_{\mu\nu}(x'),\psi_{\lambda\tau}(x''))|0\rangle$$

$$= \varepsilon_{\sigma\rho\mu\nu}\varepsilon_{\lambda\tau\xi\eta}[id_{\mu\nu\lambda\tau}(\partial')\Delta_c(x'-x'')] \ . \qquad (3.62)$$

Therefore,

$$S^{(2)} = G^2\iint d^4x \, d^4x'' \partial_\sigma J_\rho(x)\varepsilon_{\sigma\rho\mu\nu}[id_{\mu\nu\lambda\tau}(\partial)\Delta_c]$$

$$\varepsilon_{\lambda\tau\xi\eta} \ \partial_\rho{}' J_\eta(x')$$

$$= G^2\int d^4x \, d^4x' \ 2i[\partial_\xi J_\eta(x) - \partial_\eta J_\xi(x)]\Delta_c(x-x')\partial'_\xi J_\eta(x')$$

$$+ \ [\frac{2i}{m^2}]\{\partial_\sigma J_\xi(x)[\partial_\eta\partial_\sigma\Delta_c(x-x')]\partial'_\xi J_\eta(x')$$

$$- \ \partial_\sigma J_\eta(x)[\partial_\xi\partial_\sigma\Delta_c(x-x')]\partial'_\xi J_\eta(x')$$

$$+ \ \partial_\xi J_\rho(x)[\partial_\eta\partial_\rho\Delta_c(x-x')]\partial'_\xi J_\eta(x')$$

$$- \ \partial_\eta J_\rho(x)[\partial_\xi\partial_\rho\Delta_c(x-x')]\partial'_\xi J_\eta(x')$$

$$- \ [\partial_\xi J_\eta(x) - \partial_\eta J_\xi(x)][\Box\Delta_c(x-x')]\partial'_\xi J_\eta(x')\}. \qquad (3.65)$$

Integrating by parts we can cast (3.65) into the form

$$S^{(2)} = G^2\int d^4x\int d^4x' \;(-2i)(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\Delta_c(x-x')J_\mu(x)J_\nu(x') \tag{3.66}$$

We observe that the characteristic term

$$(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)$$

of a gauge-independent theory appears in the S-matrix and that all terms involving $(m^2)^{-1}$ as a coefficient have disappeared so we can take the $m \to 0$ limit for such processes.

3.4 The Relationship to the Usual Minimal Coupling of the Vector Field to the Dirac Field

We have shown that although we do not have such a simple connection between gauge-invariance and the interaction Lagrangian as in Quantum Electrodynamics our gauge principle does indeed restrict the form of the interaction. We will now show that our interaction is equivalent to the usual "minimal" coupling of the vector field to a Dirac field when the gauge is fixed.

The equations of motion we have obtained for the tensor field interacting with the fermion field are

$$(\Box - \mu^2)\psi_{\mu\nu}(x) = J_{\mu\nu}(x) = G\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma(\bar{B}(x)\gamma_\rho B(x)) \tag{3.67}$$

$$-(\gamma_\mu\partial_\mu + m)B(x) = -G\,\varepsilon_{\mu\nu\sigma\rho}(\partial_\sigma\psi_{\mu\nu})\gamma_\rho B(x) \ . \tag{3.68}$$

If we make the identification

$$\psi_{\mu\nu}(x) = \frac{i}{\sqrt{2}\,\mu}\,\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma V_\rho \tag{3.69}$$

where the factor $(\mu)^{-1}$ has been introduced so as to make the dimensions of V_ρ the same as those of the Proca field, the equation of motion (3.67) becomes

$$(\Box - \mu^2)\psi_{\mu\nu} = (\Box - \mu^2)\,\frac{i}{\sqrt{2}\,\mu}\,\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma V_\rho$$

$$= G\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha(\bar{B}(x)\gamma_\beta B(x))$$

$$(\Box - \mu^2)\,\frac{1}{\sqrt{2}\,\mu}\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma V_\rho$$

$$= -iG\varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma(\bar{B}(x)\gamma_\rho B(x)) \ . \tag{3.70}$$

That is,

$$(\Box - \mu^2)(\partial_\alpha V_\beta - \partial_\beta V_\alpha) = -i(\sqrt{2}\mu G)[\partial_\alpha(\bar{B}\gamma_\beta B) - \partial_\beta(\bar{B}\gamma_\alpha B)] \tag{3.71}$$

If we put

$$g \equiv \sqrt{2}\,\mu\,G \tag{3.72}$$

and

$$J_\rho(x) = -ig(\bar{B}(x)\gamma_\rho B(x)) \tag{3.73}$$

then we can write

$$(\Box - \mu^2)(\partial_\alpha V_\beta - \partial_\beta V_\alpha) = (\partial_\alpha J_\beta - \partial_\beta J_\alpha) \ . \qquad (3.74)$$

This is exactly the equation of motion suggested by Coombes [1968] to describe the photon; the $\mu \to 0$ limit being taken at the end of any practical calculation.

The gauge can now be fixed so that the vector field obeys the Proca equation. To do this we note that equation (3.74) implies

$$(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu - \mu^2\delta_{\mu\nu})V_\nu - J_\mu = \partial_\mu F(x) \qquad (3.75)$$

for any arbitrary function $F(x)$. As the theory is invariant under the gauge transformation

$$V_\mu \to V_\mu' = V_\mu + \partial_\mu \Lambda \qquad (3.76)$$

we can always find some V_μ such that $\partial_\mu F(x) = 0$.

To see this, let us make a gauge transformation of the left hand side of (3.75):

$$\begin{aligned}(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu - \mu^2\delta_{\mu\nu})V_\nu' - J_\mu \\ &= (\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu - \mu^2\delta_{\mu\nu})V_\nu - J_\mu - \mu^2\partial_\mu\Lambda(x) \\ &= \partial_\mu F(x) - \mu^2\partial_\mu\Lambda(x) \ . \qquad (3.77)\end{aligned}$$

Thus if we pick $\Lambda(x)$ so that

$$\partial_\mu F(x) - \mu^2 \partial_\mu \Lambda(x) = 0 \quad , \tag{3.78}$$

we obtain

$$(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu - \mu^2\delta_{\mu\nu})V_\nu(x) = J_\mu(x) \quad . \tag{3.79}$$

In this gauge

$$\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma\psi_{\mu\nu}(x) = \left(\frac{i\sqrt{2}}{\mu}\right) \varepsilon_{\mu\nu\sigma\rho}\partial_\sigma\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha V_\beta$$

$$= (i/\sqrt{2}\mu)(\Box\delta_{\sigma\rho} - \partial_\sigma\partial_\rho)V_\rho \quad . \tag{3.80}$$

Making use of (3.79) we find that

$$\varepsilon_{\mu\nu\sigma\rho}\partial_\sigma\psi_{\mu\nu} = \frac{i\sqrt{2}}{\mu} (\mu^2 V_\rho + J_\rho) \quad . \tag{3.81}$$

The equation of motion for the fermion field can be re-written using (3.81):

$$-(\gamma_\mu\partial_\mu + m)B(x) = -G \frac{i\sqrt{2}}{\mu} [\mu^2 V_\rho + J_\rho]\gamma_\rho B(x)$$

$$= -igV_\rho\gamma_\rho B - \frac{ig}{\mu^2} J_\rho\gamma_\rho B$$

$$= -igV_\rho\gamma_\rho B + \frac{g^2}{\mu^2} (i\bar{B}(x)\gamma_\rho B(x))i\gamma_\rho B(x) \quad . \tag{3.82}$$

Insisting on the gauge-independence of the spin-one field forced the form of the interaction Lagrangian to be

$$\mathcal{L}_{int} = G\varepsilon_{\mu\nu\sigma\rho}(\partial_\sigma\psi_{\mu\nu})(\bar{B}\gamma_\rho B) \ . \tag{3.83}$$

We have shown that fixing the gauge in such a way as to regain the Proca type field equations for the boson we are led to the usual form of "minimal" coupling plus a four-fermion current-current interaction. What is also interesting is that we have a definite relationship between the two coupling constants that appear in the theory.

We have calculated the effective interaction Hamiltonian for the system described by the equations (3.67) and (3.68) and have found that it is just

$$(\mathcal{H}_{int})_{effective} = -\mathcal{L}_{int}$$

which in this gauge is

$$(\mathcal{H}_{int})_{effective} = V_\rho(x)J_\rho(x) + \frac{1}{\mu^2} J_\rho J_\rho \ . \tag{3.84}$$

In this gauge we have the possibility of a bound state of a fermion and antifermion pair arising through the current-current interaction. This possibility will be discussed in chapter five when we calculate the propagator for the meson field. But first we will digress to discuss a slightly different problem involving a four-fermion interaction.

CHAPTER 4

FOUR-FERMION INTERACTIONS

The interaction we considered in the last chapter raises a very basic problem of Quantum Field Theory: The renormalization of theories in which bound states appear, or more generally in which more than one mass is associated with any one Heisenberg field. In this chapter we will look at this problem and, in particular, investigate a model in which the usual Yukawa pseudoscalar meson interaction with the Dirac field is modified by a four-fermion term of the type that appeared in the vector meson interaction.

4.1 Introduction

A four-fermion interaction term corresponds to a Feynman diagram in which four fermion lines meet at a point. If two of these go on to another four fermion junction then the fermion-antifermion pair that propagates between these two vertices will behave as a composite particle with integer spin and zero fermion number. We will have a bound state which, in general, will be associated with some mass different from the mass parameters that enter the original Lagrangian.

The multi-mass aspect of the meson propagator enters our theory at another level. The asymptotic vector field obeys the equation

$$(\Box - m^2)(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)V_\mu = 0 \tag{4.1}$$

which suggests that both a mass m and a mass zero component will be present. This equation is also the equation that describes the vector field that appears in the spontaneous breakdown of symmetry in the Nambu model [Nambu et al., 1961] as has been shown by Aurilia and Takahashi [1971]. If this were not an asymptotic field equation but rather a free field equation derived directly from a Lagrangian we would expect the propagator to have the momentum dependence

$$\frac{1}{k^2+m^2} \cdot \frac{1}{k^2} = -\frac{1}{m^2}\left[\frac{1}{k^2+m^2} - \frac{1}{k^2}\right] \tag{4.2}$$

where the residues at the two poles would have opposite signs. If this were the case, we would not be able to simultaneously normalize both the mass m and the mass zero states. In our theory, equation (4.1) is not directly derivable from a Lagrangian so it is very difficult to see how to proceed with the analysis of this problem.

In this chapter we will examine instead the case of a Dirac field interacting with a pseudo-scalar meson through a Yukawa interaction to which is added a four-fermion current-current term. In this theory a fermion-antifermion bound state appears allowing us to examine the problem of renormalization under these circumstances.

To see exactly how a problem arises when a field is associated with more than one mass state, let us consider the features of the one-body propagator for a spinor field which has a finite number of stable states with masses.*

$$m^{(1)}, m^{(2)}, \ldots\ldots m^{(n)} \quad (m^{(1)} < m^2 < \ldots. < m^{(n)}) .$$

We may expect that the propagator obeys the equation

$$i(i\,\gamma.p + \mu + M(-i\gamma.p))G(p) = 1 \tag{4.3}$$

where the masses are the roots of the equation

$$\alpha - \mu - M(\alpha) = 0 \quad . \tag{4.4}$$

We will assume that these mass values correspond to simple roots of (4.4) and that they are all in the domain

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* This part of the introduction follows that given by Umezawa, 1956, Chapter 18.

$$\alpha \leq c \quad . \tag{4.5}$$

The operator

$$h(-i\gamma.p) \equiv (-i\gamma.p - \mu - M(-i\gamma.p)) \tag{4.6}$$

can be written in the form

$$h(-i\gamma.p) = a(-i\gamma.p) \prod_{j=1}^{n} (-i\gamma.p - m^{(j)}) \tag{4.7}$$

where $a(\alpha)$ has no zeros for $\alpha < c$.

For each mass we can define a constant $Z^{(j)}$ by

$$(Z^{j})^{-1} = \frac{\partial}{\partial\alpha} (\alpha - M(\alpha)) \Big|_{\alpha=m^{(j)}} \tag{4.8}$$

$$= a(m^{j}) \prod_{j\neq k} (m^{(j)} - m^{(k)}) \quad . \tag{4.9}$$

If we use (4.7) and (4.9) it can be shown that

$$G(p) = \sum_{j} \frac{iZ^{(j)}}{-i\gamma.p - m^{(j)} + i\varepsilon} \quad \frac{a(m^{j})}{a(-i\gamma.p)} \tag{4.10}$$

Thus, $Z^{(j)}$ is just the renormalization constant for the propagator of the particle with mass $m^{(j)}$. Since $a(\alpha)$ has no zeros for $\alpha < c$ it has no zeros for $\alpha < m^{(n)}$. Clearly if $a(\alpha)$ has no singularities in this domain also then it is either positive or negative definite. It follows from (4.9) that the sign of $Z^{(j)}$ alternates with increasing j.

$$\frac{Z^{(j)}}{Z^{(j+2m+1)}} < 0 \quad , \quad m = \text{integer} . \tag{4.11}$$

In other words, half the Z's are positive while the other half are negative.

However, Umezawa has shown that $Z^{(j)}$ can also be written as

$$Z^{(j)} = \langle 0|\psi(p^{(j)})|p^{(j)}\rangle\langle p^{(j)}|\bar{\psi}(p^{(j)})|0\rangle \tag{4.12}$$

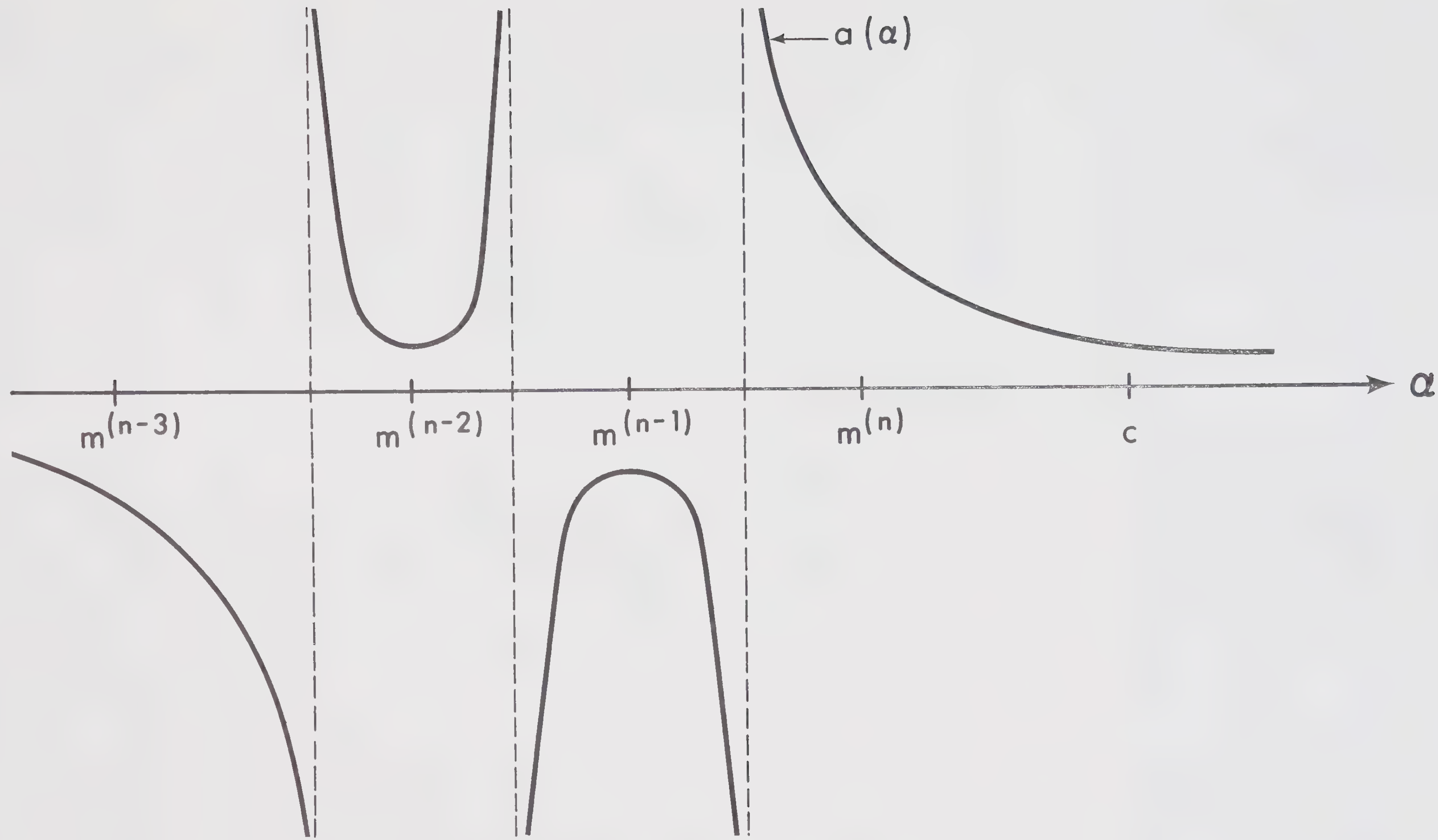
in which case we are led to the conclusion that $\langle 0|\psi|p^{(j)}\rangle$ is the conjugate of $\langle p^{(j)}|\bar{\psi}|0\rangle$ only for those j for which $Z^{(j)}$ is positive! The Hamiltonian can then no longer be hermitian and we must expect a non-unitary S-matrix.

If we are to avoid these conclusions, and thus the entire breakdown of our theory, our assumptions must be incorrect. In particular, we must assume that $a(\alpha)$ does have singularities in the domain $\alpha < c$. What is more, it must have a singular point between each root $m_{(j)}$ and the next, $m_{(j+1)}$, as in Figure 1.

It is not at all obvious that this is the kind of behaviour we should expect to get from a realistic calculation. We will show that these are exactly the features we find in $a(\alpha)$ for the case of the Dirac field interacting with a pseudo-scalar meson through a modified Yukawa interaction, at least in the pair-approximation.

Figure (1)

A sketch graph of the function $a(\alpha)$ which appears in equation (4.7). The graph is to illustrate the necessary distribution of singular points in relation to the mass values $m^{(j)}$.


a (α)
α
m(n-3)
m(n-2)
m(n-1)
m(n)
c

In the Lee Model (Lee, 1954) we know that states with negative norm, the so-called ghosts, appear. We will look at this model firstly as an illustration of the techniques we shall use for the Yukawa case but also because the model can be solved without recourse to an approximation.

4.2 The Lee Model

It will be assumed that the general features of the Lee Model are well known to the reader. All that is, in fact, necessary to understand this section is that the model describes a fermion that can exist in two states, the V-state and the N-state. When in the V-state it can emit a boson to transform itself into the N-state. The N-state cannot emit a boson but can absorb one to return to the V-state. Symbolically this can be summarized by the equation

$$V \rightleftharpoons N + \phi \; . \tag{4.13}$$

The process

$$N \rightleftharpoons V + \phi$$

is not allowed. In effect this means no fermion anti-particle exists. This theory is non-local and non-causal but it is solvable. It's simplicity lies in the fact that only the V-state mass is renormalized by the

interaction. The bare N mass and the ϕ mass coincide with their physical values.

As the model lacks an antiparticle corresponding to the V and N fields it is intrinsically non-relativistic, so we can use the Schrödinger equations:

$$(\partial_4 + \kappa_{obs})\psi_V = -g\psi_N\,\phi^{(+)} + \delta\kappa\psi_V \quad , \tag{4.14}$$

$$\equiv \eta(x) \quad ; \tag{4.15}$$

$$(\partial_4 + M)\psi_N = -g\psi_V\phi^{(-)} \tag{4.16}$$

$$(i\partial_t - \omega(\nabla^2))\phi^{(+)}(x) = g\left(\frac{1}{2\omega(\nabla^2))}\right)\bar{\psi}_N\psi_V \tag{4.17}$$

$$(-i\partial_t - \omega(\nabla^2))\phi^{(-)}(x) = g\left(\frac{1}{2\omega(\nabla^2)}\right)\bar{\psi}_V\psi_N \tag{4.18}$$

where $\omega(\underset{\sim}{k})$ is the energy of the boson, i.e. $(\underset{\sim}{k}^2+m^2)^{\frac{1}{2}}$ if the mass of the ϕ particle is m. The non-local aspects of the theory are clearly illustrated by the splitting of the ϕ field into its positive and negative frequency parts, $\phi^{(+)}$ and $\phi^{(-)}$. This involves an integration over all space.

For the free fields we can write the Green's functions as:

$$\langle 0|T(\psi_V(x),\bar{\psi}_V(x'))|0\rangle = -i\ S_c^{(V)}(x-x') \quad ; \tag{4.19}$$

$$<0|T(\phi^{(+)}(x),\phi^{(-)}(x'))|0> = i\Delta_c(x-x') \quad ; \tag{4.20}$$

$$<0|T(\psi_N(x),\bar{\psi}_N(x'))|0> = -iS_c^{(N)}(x-x') \quad . \tag{4.21}$$

The quantities on the left hand sides of these definitions obey the equations:

$$(\partial_4 + \kappa_{obs})\, S_c^{(V)}(x-x') = \delta^4(x-x') \quad ; \tag{4.22}$$

$$(\partial_4 + M)\, S_c^{(N)}(x-x') = \delta^4(x-x') \quad ; \tag{4.23}$$

$$(i\partial_t - \omega(\nabla^2))\Delta_c(x-x') = \frac{1}{2\omega(\nabla^2)}\, \delta^4(x-x') \quad . \tag{4.24}$$

Explicitly we have

$$S_c^{(V)}(x-x') = \frac{1}{(2\pi)^4}\int d^4p\, \frac{1}{\kappa_{obs}-p_o-i\varepsilon}\, e^{ip(x-x')} \quad , \tag{4.25}$$

$$S_c^{(N)}(x-x') = \frac{1}{(2\pi)^4}\int d^4p\, \frac{1}{M - p_o - i\varepsilon}\, e^{ip(x-x')} \quad , \tag{4.26}$$

$$\Delta_c(x-x') = \frac{1}{(2\pi)^4}\int d^4k\, \frac{1}{2\omega(\underset{\sim}{k})(k_o-\omega_k+i\varepsilon)}\, e^{ik(x-x')} . \tag{4.27}$$

The first stage in our investigation of this model is to determine the mass correction $\delta\kappa$. To do this we must calculate the Green's function for the V-particle when the interaction is present. We denote this by $S_c^{(V)'}(x-x')$.

$$-iS_c^{(V)'}(x-x') \equiv \langle 0|T(\psi_V(x),\psi_V(x'))|0\rangle \quad . \tag{4.28}$$

Using the equation of motion (4.14) we see that

$$(\partial_4 + \kappa_{obs})\langle 0|T(\psi_V(x),\bar{\psi}_V(x'))|0\rangle$$

$$= -i\delta^4(x-x') + \langle 0|T(\eta(x),\bar{\psi}_V(x'))|0\rangle \quad . \tag{4.29}$$

To simplify our future expressions we define a function $\Sigma(q_o)$ by the equation

$$S_c^{(N)}(x-x')\Delta_c(x-x') = -\frac{i}{(2\pi)^4}\int d^4q\; e^{iq(x-x')}\Sigma(q_o) \quad . \tag{4.30}$$

We can use the free particle propagators $S_c^{(N)}(x-x')$ and $\Delta_c(x-x')$ to re-express the right hand as these are not renormalized by the interaction. Thus,

$$S_c^{(N)}(x-x')\Delta_c(x-x')$$

$$= \frac{1}{(2\pi)^8}\int d^4p\int d^4k\; \frac{1}{(M-p_o-i\varepsilon)}\;\frac{1}{2\omega(k)}\;\frac{1}{k_o-\omega(k)+i\varepsilon}\;\times$$

$$e^{ip(x-x')}\; e^{ik(x-x')}$$

$$= -\frac{i}{(2\pi)^4}\int d^4p\int d^4k\; \frac{\pi}{(2\pi)^4}\;\frac{1}{(M-p_o-i\varepsilon)}\;\frac{1}{2\omega(k)}\;\times$$

$$\delta(k_o-\omega(k))\; e^{i(p+k)(x-x')} \quad . \tag{4.31}$$

If we put q = p+k we get

$$S_c^{(N)}(x-x')\Delta_c(x-x') = -\frac{i}{(2\pi)^4}\int d^4q\ e^{iq(x-x')}$$

$$\times\ \frac{\pi}{(2\pi)^4}\int d^3k\ \frac{1}{2\omega(k)}\ \frac{1}{(M-q_o+\omega(k)-i\varepsilon)}\quad , \tag{4.32}$$

so we can see that $\Sigma(q_o)$ can be written as

$$\Sigma(q_o) = \frac{\pi}{(2\pi)^4}\int \frac{d\tilde{k}}{2\omega(k)}\ \frac{1}{(M-q_o+\omega(k)-i\varepsilon)}\quad . \tag{4.33}$$

Hence,

$$<0|T(\eta(x),\bar{\psi}_V(x'))|0> = -ig<0|T(\psi_N(x)\phi^{(+)}(x),\bar{\psi}_V(x'))|0>$$

$$+\ \delta\kappa<0|T(\psi_V(x),\bar{\psi}_V(x')|0>$$

$$= -\frac{ig^2}{(2\pi)^4}\int d^4q\ e^{iq(x-x')}\ \Sigma(q_o)\ S_c^{(V)'}(q)$$

$$-\ i\ \frac{\delta\kappa}{(2\pi)^4}\int d^4q\ e^{iq(x-x')}\ S_c^{(V)'}(q)\quad . \tag{4.34}$$

In the final equation we have inserted the Fourier transform of $S_c^{(V)'}(x-x')$ defined by

$$S_c^{(V)'}(x-x') = \frac{1}{(2\pi)^4}\int d^4q\ e^{iq(x-x')}\ S_c^{(V)'}(q)\ , \tag{4.35}$$

and have made use of (4.30) and the fact that

$$<0|T(\psi_N(x)\phi^{(+)}(x),\bar{\psi}_V(x'))|0>$$

$$= -g\int d^4y\ S_c^{(N)}(x-y)\Delta_c(x-x')S_c^{(V)'}(y-x')$$

$$= \frac{ig}{(2\pi)^4}\int d^4q\ e^{iq(x-x')}\ \Sigma(q_o)S_c^{(V)'}(q)\ . \qquad (4.36)$$

Then (4.29) implies that

$$(\kappa_{obs} - q_o)S_c^{(V)'}(q) = 1 + g^2\ \Sigma(q_o)S_c^{(V)'}(q)$$

$$+ \delta\kappa\ S_c^{(V)'}(q)\ . \qquad (4.37)$$

When $q_o = \kappa_{obs}$ the right hand side reduces to a constant so we must require that

$$\delta\kappa = -\ g^2\ \Sigma(\kappa_{obs})\ . \qquad (4.38)$$

A more rigorous proof of this relationship can be obtained as follows:

$$(\partial_4 + \kappa_{obs})<0|T(\psi_V(x),\bar{\psi}_V(x'))|0>(-\overleftarrow{\partial}'_4 + \kappa_{obs})$$

$$= <0|T(\eta(x),\eta(x'))|0> - i(\partial_4 + \kappa_{obs} + \delta\kappa)\delta^4(x-x')\ .$$

(4.39)

To get this result we have used (4.15) and its hermitian conjugate, defining $\bar{\eta}(x)$ in the obvious way, i.e.

$$<0|T(\eta(x),\bar{\eta}(x'))|0>$$

$$= g^2<0|T(\psi_N(x)\phi^+(x),\bar{\psi}_N(x')\phi^-(x))|0>$$

$$-g\delta\kappa<0|T(\psi_V(x),\bar{\psi}_N(x')\phi^{(-)}(x'))|0>$$

$$-g\delta\kappa<0|T(\psi_N(x)\phi^{(+)}(x),\bar{\psi}_V(x'))|0>$$

$$+ (\delta\kappa^2)<0|T(\psi_V(x),\bar{\psi}_V(x'))|0> \quad . \tag{4.40}$$

The only term we cannot express immediately in terms of the functions we have already defined is that proportional to g^2. However,

$$<0|T(\psi_N(x)\phi^+(x),\bar{\psi}_V(x'))|0>(-\overleftarrow{\partial}_4' + \kappa_{obs})$$

$$= <0|T(\psi_N(x)\phi^{(+)}(x),\bar{\eta}(x'))|0> \tag{4.41}$$

$$= -g<0|T(\psi_N(x)\phi^{(+)}(x),\bar{\psi}_N(x')\phi^{(-)}(x'))|0>$$

$$+ \delta\kappa<0|T(\psi_N(x)\phi^{(+)}(x),\bar{\psi}_V(x'))|0> \quad . \tag{4.42}$$

The right hand side can be expressed in terms of $\Sigma(q_o)$ using (4.36) so we can derive an expression for $<0|T(\psi_N(x)\phi^+(x),\bar{\psi}_N(x')\phi^{(-)}(x')|0>$. If we do this we find that

$$<0|T(\eta(x),\bar{\eta}(x'))|0>$$

$$= -\frac{i}{(2\pi)^4}\int d^4q\ e^{iq(x-x')}[g^2\Sigma(q_o)+\delta\kappa g^2\Sigma(q_o)S_c^{(V)'}(q)$$

$$+ (\delta\kappa)^2 S_c^{(V)'}(q) + g^2\Sigma(q_o)\{g^2\Sigma(q_o)S_c^{(V)'}(q)$$

$$+ \delta\kappa\ S_c^{(V)'}(q)\}] \ . \tag{4.43}$$

The substitution of these results into equation (4.39) gives

$$(\kappa_{obs} - q_o)\ S_c^{(V)'}(q)(\kappa_{obs} - q_o)$$

$$= g^2\Sigma(q_o) + 2\delta\kappa g^2 S_c^{(V)'}(q) + (\delta\kappa)^2 S_c^{(V)'}(q)$$

$$+ g^4\Sigma(q_o)\Sigma(q_o)S_c^{(V)'}(q) + (\kappa_{obs} - q_o) + \delta\kappa \tag{4.44}$$

$$= (g^2\Sigma(q_o) + \delta\kappa) + (\kappa_{obs} - q_o)$$

$$+ (\delta\kappa + g^2\Sigma(q_o))^2\ S_c^{(V)'}(q) \ . \tag{4.45}$$

The right hand side of this equation vanishes when $q_o = \kappa_{obs}$ so we obtain, as before,

$$\delta\kappa = -g^2\ \Sigma(\kappa_{obs}) \ . \tag{4.46}$$

Let us write

$$g^2\Sigma(q_o) - g^2\Sigma(\kappa_{obs}) \equiv (\kappa_{obs} - q_o)a(q_o) \tag{4.47}$$

where $a(q_o)$ is given by the definition of $\Sigma(q_o)$ to be

$$a(q_o) = -\frac{g^2\pi}{(2\pi)^4}\int \frac{d\underset{\sim}{k}}{2\omega(k)} \frac{1}{(M - q_o + \omega(k) - i\varepsilon)} \cdot \frac{1}{M - \kappa_{obs} + \omega(k)} . \tag{4.48}$$

Therefore,

$$(\kappa_{obs} - q_o)S_c^{(V)'}(q)(\kappa_{obs} - q_o)$$

$$= (\kappa_{obs} - q_o)[a(q_o)+1] + (\kappa_{obs} - q_o)^2 \; a^2(q_o)S_c^{(V)'}(q) . \tag{4.49}$$

Following the general theory we outlined in the first section of this chapter we can define the renormalization constant for the propagator of particle with mass κ_{obs}, $Z^{(1)}$, by the equation

$$(Z^{(1)})^{-1} = 1 + g^2 \left(\frac{\partial\Sigma(q_o)}{\partial q_o}\right)_{q_o=\kappa_{obs}} . \tag{4.50}$$

Hence,

$$Z^{(1)} = a(\kappa_{obs}) + 1 + Z^{(1)} \; a^2(\kappa_{obs}) .$$

or

$$Z^{(1)} = \frac{1 + a(\kappa_{obs})}{1 - a^2(\kappa_{obs})}$$

$$= \frac{1}{1 - a(\kappa_{obs})} \tag{4.51}$$

using the definition of $a(q_o)$, (4.47), we find that

$$(Z^{(1)})^{-1} = 1 - a(\kappa_{obs})$$

$$= 1 + \frac{g^2\pi}{(2\pi)^4} \int \frac{d^3k}{2\omega(k)} \frac{1}{(M-\kappa_{obs}+\omega(k))^2} \quad . \tag{4.52}$$

This can be rewritten as

$$Z^{(1)} = 1 - g^2_{obs} \frac{\pi}{(2\pi)^4} \int \frac{d^3k}{2\omega(k)} \frac{1}{(M-\kappa_{obs}+\omega(k))^2} \tag{4.53}$$

where

$$g^2_{obs} \equiv Z^{(1)} g^2 \quad , \tag{4.54}$$

g_{obs} being the physically "observed" coupling constant.

We are now in a position to examine the features of the propagator $S_c^{(V)'}(q)$. Equations (4.45) and (4.37) both lead to the result

$$S_c^{(V)'}(q) = \frac{1}{\kappa_{obs} - q_o} \cdot \frac{1}{1 - a(q_o)} \quad . \tag{4.55}$$

Thus the renormalized propagator, $S_R^{(V)}(q)$, which is defined by

$$S_R^{(V)}(q) = (Z^{(1)})^{-1} S_c^{(V)'}(q) \tag{4.56}$$

is

$$S_R^{(V)}(q) = \frac{1}{(\kappa_{obs} - q_o)} \cdot \frac{1}{Z^{(1)} - Z^{(1)} a(q_o)} \; . \tag{4.57}$$

The object of the renormalization procedure we have followed has been to preserve the pole in the renormalized propagator at the "observed" mass of the V-particle. The propagator given by (4.57) certainly has such a pole but there is also the possibility of a second pole if for some value of q_o, say M_b,

$$Z^{(1)} - Z^{(1)} a(M_b) = 0 \; . \tag{4.58}$$

This would correspond to a bound state.

To see if this is possible, let us rewrite (4.58) using the definitions of $Z^{(1)}$ and $a(q_o)$. Then

$$Z^{(1)} - Z^{(1)} a(q_o) = 1 + \frac{g_{obs}^2 \pi}{(2\pi)^4} \int \frac{d^3k}{2\omega(k)} \frac{q_o - \kappa_{obs}}{(M - \kappa_{obs} + \omega(k))^2}$$

$$\cdot \frac{1}{(M - q_o + \omega(k) - i\varepsilon)} \; . \tag{4.59}$$

Therefore,

$$Z^{(1)} - Z^{(1)}a(M_b) = 1 + (M_b - \kappa_{obs}) \frac{g^2_{obs}\pi}{(2\pi)^4} \int \frac{d^3k}{2\omega(k)}$$

$$\cdot \frac{1}{(M-\kappa_{obs}+\omega(k))^2} \frac{1}{(M-M_b+\omega(k)-i\varepsilon)} \quad . \tag{4.60}$$

The second term on the right hand side of this equation can be negative if $M_b<K_{obs}<M+M_b$, so we do indeed have the possibility of a bound state. However, it is easily shown that

$$S_R^{(V)}(q) = \left(\frac{1}{(\kappa_{obs}-q_o)} - \frac{1}{(M_b- q_o)}\right) \frac{b(q_o)}{M_b- \kappa_{obs}} \tag{4.61}$$

where

$$Z^{(1)} - Z^{(1)}a(q_o) \equiv (M_b- q_o)\; b^{-1}(q_o) \quad . \tag{4.62}$$

That is

$$b^{-1}(q_o) = - \frac{g^2_{obs}\pi}{(2\pi)^4} \int \frac{d\tilde{k}}{2\omega(k)}$$

$$\cdot \frac{1}{(M+\omega(k)-\kappa_{obs})(M-M_b+\omega(k)(M-q_o+\omega(k))} \quad . \tag{4.63}$$

Therefore

$$\frac{b(\kappa_{obs})}{M_b-\kappa_{obs}} > 0 \tag{4.64}$$

and

$$\frac{b(M_b)}{M_b - \kappa_{obs}} > 0 \tag{4.65}$$

which means that the state with mass M_b has a negative norm and is referred to as a "ghost".

The appearance of this difficulty can be traced back to a basic inconsistence in the argument: Let us re-examine the definition of g_{obs}. From equation (4.52) we see that

$$\frac{1}{g_{obs}^2} = \frac{1}{g^2} + \frac{\pi}{(2\pi)^4} \int \frac{d^4k}{2\omega(k)} \frac{1}{(M-\kappa_{obs}+\omega(k))^2} \quad . \tag{4.66}$$

The left hand side must be finite to agree with our identification of g_{obs} with the physical coupling constant. The second term on the right hand side is however positive and infinite so g^2 must be negative! It follows that the Hamiltonian is not hermitian and the S-matrix non-unitary.

We will not examine this model further. The results we have obtained are well known. They are only included here to illustrate the methods we will use for the Yukawa case and to show how the problem of negative norms arise in theories involving bound states.

The calculation for the Lee Model can be performed including a $\bar{\psi}_N\psi_N\phi^{(+)}\phi^{(-)}$ interaction. In this

case the problem of negative norms does not arise for exactly the same reason that it does not appear in the Yukawa case when the four-fermion interaction is taken into account.

4.3 The Modified Yukawa Interaction

Now let us turn our attention to a physically more interesting model. We will consider the case of a fermion field coupled to a pseudo-scalar meson through a Yukawa type interaction to which we will add a four-fermion interaction. The total Lagrangian density is then

$$\begin{aligned}\mathcal{L} = &-\bar{\psi}(x)(\gamma_\mu \partial_\mu + m)\psi(x) + if\bar{\psi}\gamma_5\psi\phi(x) \\ &- \frac{1}{2}\left(\partial_\mu\phi(x)\partial_\mu\phi(x) + \mu^2\phi^2(x)\right) \\ &- g(\bar{\psi}\gamma_5\psi)^2 + \frac{1}{2}\Delta\mu^2\phi^2(x) + \delta m\bar{\psi}(x)\psi(x) \quad . \qquad (4.67)\end{aligned}$$

Variation with respect to $\bar{\psi}(x)$ and $\phi(x)$ yields the Euler-Lagrange equations

$$\begin{aligned}(\gamma_\mu\partial_\mu + m)\psi(x) = &\; if\gamma_5\psi(x)\phi(x) + \delta m\psi(x) \\ &+ 2gi\gamma_5\psi(x)(i\bar{\psi}(x)\gamma_5\psi(x)) \quad . \qquad (4.68)\end{aligned}$$

$$\begin{aligned}(\Box - \mu^2)\phi(x) &= -if\bar{\psi}(x)\gamma_5\psi(x) - \Delta\mu^2\phi(x) \\ &= j(x) \quad . \qquad (4.69)\end{aligned}$$

Using the usual notation, we will define the unrenormalized propagator for the meson field when the interaction described by (4.67) is present to be

$$iD_c'(x-x') = \langle 0|T(\phi(x),\phi(x'))|0\rangle \quad . \tag{4.70}$$

The first stage in the investigation of the features of this propagator is to set up the Dyson's equations and find some way of calculating the mass correction $\Delta\mu^2$. To do this, let us proceed in the same way as we did for the Lee Model. Using the equation of motion (4.69) we see that

$$(\Box - \mu^2)\langle 0|T(\phi(x),\phi(x'))|0\rangle$$

$$= \langle 0|T(j(x),\phi(x'))|0\rangle + i\delta^4(x-x') \tag{4.71}$$

$$=-if\langle 0|T(\bar{\psi}\gamma_5\psi(x),\phi(x'))|0\rangle$$

$$- \Delta\mu^2\langle 0|T(\phi(x),\phi(x'))|0\rangle + i\delta^4(x-x') \quad . \tag{4.72}$$

We are now confronted with the problem of evaluating the term proportional to f in (4.72). First let us define the vertex function Γ_5 by the relationship

$$\langle 0|T(\phi(x),\psi_\alpha(y),\bar{\psi}_\beta(z))|0\rangle$$

$$= if \int d^4x'd^4y'd^4z'S_c'^{\alpha\alpha'}(y-y')\Gamma_5^{\alpha'\beta'}(y'-x';x'-z')$$

$$\times \quad S_c'^{\beta'\beta}(z'-z)D_c'(x-x') \quad , \tag{4.73}$$

where $S_c'(x-x')$ is the propagator for the fermion field. This is, of course, unknown at this stage as, unlike the Lee Model case, it will be renormalized by the interaction. However, we can at least formally calculate $\langle 0|T(\bar{\psi}(x)\gamma_5\psi(x),\phi(x'))|0\rangle$ using (4.73)

$$\langle 0|T(\bar{\psi}(x)\gamma_5\psi(x),\phi(x'))|0\rangle$$

$$= -\frac{if}{(2\pi)^8}\int d^4k\, e^{ik(x-x')}\int d^4p$$

$$\times \mathrm{Tr}[\gamma_5 S_c'(p)\Gamma_5(p,p-k)S_c'(p-k)]D_c'(k^2) \quad . \qquad (4.74)$$

The quantities $S_c'(p)$ and $D_c'(p)$ are just the Fourier transforms of the propagators:

$$S_c'(x-x') = \frac{1}{(2\pi)^4}\int d^4p\, e^{ip(x-x')}S_c'(p) \;; \qquad (4.75)$$

$$D_c'(x-x') = \frac{1}{(2\pi)^4}\int d^4k\, e^{ik(x-x')}D_c'(k) \quad . \qquad (4.76)$$

We have also written $D_c'(k) \equiv D_c'(k^2)$, recognizing that it can only be a function of k^2.

To simplify the equations we can define a function $\pi^*(k^2)$ which plays the role of $\Sigma(q_o)$ that appeared in the Lee Model.

$$\pi^*(k^2) \equiv -\frac{i}{(2\pi)^4}\int d^4p\, \mathrm{Tr}[\gamma_5 S_c'(p)\Gamma_5(p;p-k)S_c'(p-k)] \quad . \qquad (4.77)$$

Therefore,

$$<0|T(\bar{\psi}(x)\gamma_5\psi(x),\phi(x'))|0>$$

$$= \frac{f}{(2\pi)^4}\int d^4k\ e^{ik(x-x')}\ \pi^*(k^2)D_c'(k^2)\ . \qquad (4.78)$$

We can now write equation (4.72) as

$$(\square - \mu^2)<0|T(\phi(x),\phi(x'))|0>$$

$$= \frac{1}{(2\pi)^4}\int d^4k\ e^{ik(x-x')}(-i)(k^2+\mu^2)D_c'(k^2)$$

$$= -\frac{if^2}{(2\pi)^4}\int d^4k\ e^{ik(x-x')}\pi^*(k^2)D_c'(k^2)$$

$$-\Delta\mu^2\ \frac{i}{(2\pi)^4}\int d^4k\ e^{ik(x-x')}D_c'(k^2) + i\delta^4(x-x')\ . \qquad (4.79)$$

This implies

$$-(k^2+\mu^2)D_c'(k^2) = -f^2\pi^*(k^2)D_c'(k^2) - \Delta\mu^2 D_c'(k^2) + 1 \qquad (4.80)$$

or

$$D_c'(k^2) = \frac{-1}{k^2+\mu^2-f^2\pi^*(k^2)-\Delta\mu^2}\ . \qquad (4.81)$$

To ensure that this exhibits a pole at the physical mass of the meson, μ, we must have

$$f^2\pi^*(-\mu^2) + \Delta\mu^2 = 0\ ,$$

i.e.

$$\Delta\mu^2 = - f^2\pi^*(-\mu^2). \tag{4.82}$$

This is the formal solution of our problem, however we are left with the task of finding an expression for $\pi^*(k^2)$. There is no obvious way of doing this in general so we must employ an approximation. The approximation we shall use is the "pair-approximation", sometimes called the "chain-approximation", in which only a fermion-antifermion pair is allowed to be created when a meson is annihilated. In this approximation $S_c'(p) \equiv S_c(p)$, i.e. the fermion propagator has the same form as the free Dirac causal propagator, so we can write[†]

$$\pi^*(k^2) = - \frac{i}{(2\pi)^4} \int d^4p \; Tr[\gamma_5 S_c'(p)\Gamma_5(p;p-k)S_c'(p-k)]$$

$$= \frac{i}{(2\pi)^4} \int d^4p \; Tr[\gamma_5 S_F(p)\Gamma_5(p;(p-k))S_F(p-k)]$$

$$= J_p(k^2) + 2g\, J_p^2(k^2) + (2g)^2 J_p^3(k^2) + \ldots . \tag{4.83}$$

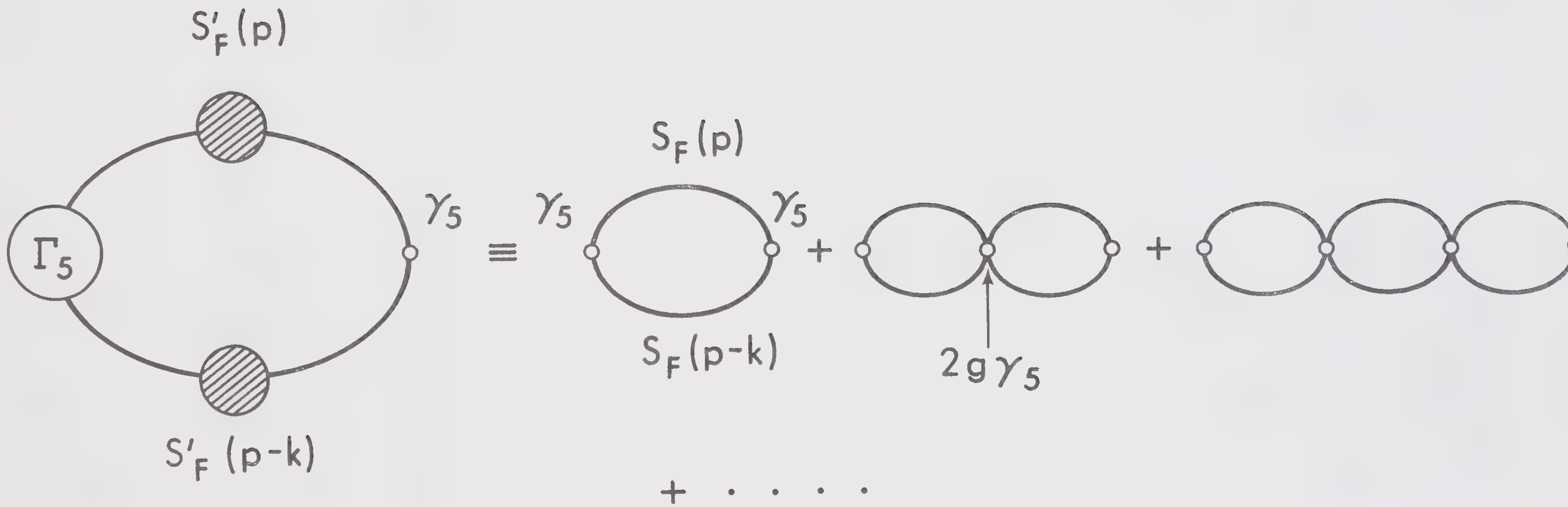
$$J_p(k^2) \equiv \frac{i}{(2\pi)^4} \int d^4p \; Tr[\gamma_5 S_F(p)\gamma_5 S_F(p-k)] \tag{4.84}$$

is the integral representing a closed fermion-antifermion

[†]This expansion of $\pi^*(k^2)$ is represented diagrammatically in Figure 2.

Figure (2)

A diagrammatical decomposition of the pseudoscalar-pseudoscalar fermion loop in the pair-approximation.



$S'_F(p)$
Γ_5
γ_5
$S'_F(p-k)$
$\equiv$
$S_F(p)$
γ_5
γ_5
$S_F(p-k)$
$+$
$2g\gamma_5$
$+$
$+ \cdot \cdot \cdot \cdot$

loop. The subscript 'p' is the usual notation used to designate that $J_p(k^2)$ is the integral for the pseudoscalar-pseudoscalar fermion loop. Summing the series (4.83) we get

$$\pi^*(k^2) = \frac{J_p(k^2)}{1 - 2g\, J_p(k^2)} \quad . \tag{4.85}$$

Thus

$$\Delta\mu^2 = - \frac{f^2 J_p(-\mu^2)}{(1 - 2g\, J_p(-\mu^2))} \quad . \tag{4.86}$$

The summation procedure we have used to derive this expression needs some further justification. If we calculate $J_p(k^2)$ we see that

$$J_p(k^2) = \frac{i}{(2\pi)^4} \int d^4p \; Tr[\gamma_5 S_F(p) \gamma_5 S_F(p-k)] \quad ,$$

$$S_F(p) = - \frac{1}{(\gamma . p - im - i\varepsilon)} \quad . \tag{4.87}$$

When we express this integral in dispersive form†, that is

$$J_p(k^2) = \frac{1}{8\pi^2} \int_{4m^2}^{\infty} dx^2 \, \frac{x^2 (1 - \frac{4m^2}{x^2})^{\frac{1}{2}}}{(x^2 + k^2)} \quad , \tag{4.88}$$

it is clear that, strictly speaking, $J_p(k^2)$ is infinite.

---

† The derivation of equation (4.88) is given in Appendix C.

For the rest of this calculation we shall introduce a cut-off Λ^2 as the upper bound on the parameter x^2. This does not remove all the problems with our summation procedure. For mathematic rigour we would have to pick Λ so that $|2gJ_p(k^2)|<1$. Instead we will look on the diagrammatical decomposition as a tool to reach the result (4.86) and justify it by showing that the same result is obtained if we use the more rigorous Bethe-Salpeter approach. This calculation reproduces the results we have already obtained so we have relegated it to Appendix D.

We are now in a position to examine the pole structure of the meson propagator, $D_c'(k^2)$. To simplify some of our expressions, let us define a function $b(k^2)$ such that

$$J_p(k^2) - J_p(-\mu^2) = b(k^2)(k^2 + \mu^2) \; . \tag{4.89}$$

Equation (4.88) gives that

$$b(k^2) = -\frac{1}{8\pi^2}\int_{4m^2}^{\Lambda^2} dx^2 \, \frac{x^2\left(1 - \frac{4m^2}{x^2}\right)^{\frac{1}{2}}}{(x^2+k^2)(x^2-\mu^2)} \; . \tag{4.90}$$

Using these results we can write the propagator as

$$D_c'(k^2) = -\frac{1}{F(k^2)} \tag{4.91}$$

where

$$F(k^2) = (k^2+\mu^2)[1 - \frac{f^2\, b(k^2)}{(1-2g\, J_p(k^2))(1-2g\, J_p(-\mu^2))}] \ . \tag{4.92}$$

Equation (4.92) clearly shows that the propagator has a pole at $k^2+\mu^2 = 0$, i.e. at the physical meson mass, and another at the fermion-antifermion bound state mass, μ_b, given by

$$\frac{f^2\, b(-\mu_b^2)}{(1-2g\, J_p(-\mu_b^2))(1-2g\, J_p(-\mu^2))} = 1 \ . \tag{4.93}$$

This result is again in agreement with the result derived from the B-S formalism given in Appendix D.

It can also be seen from equation (4.92) or more directly from (4.81) that

$$\frac{\partial F(k^2)}{\partial k^2} > 0 \tag{4.94}$$

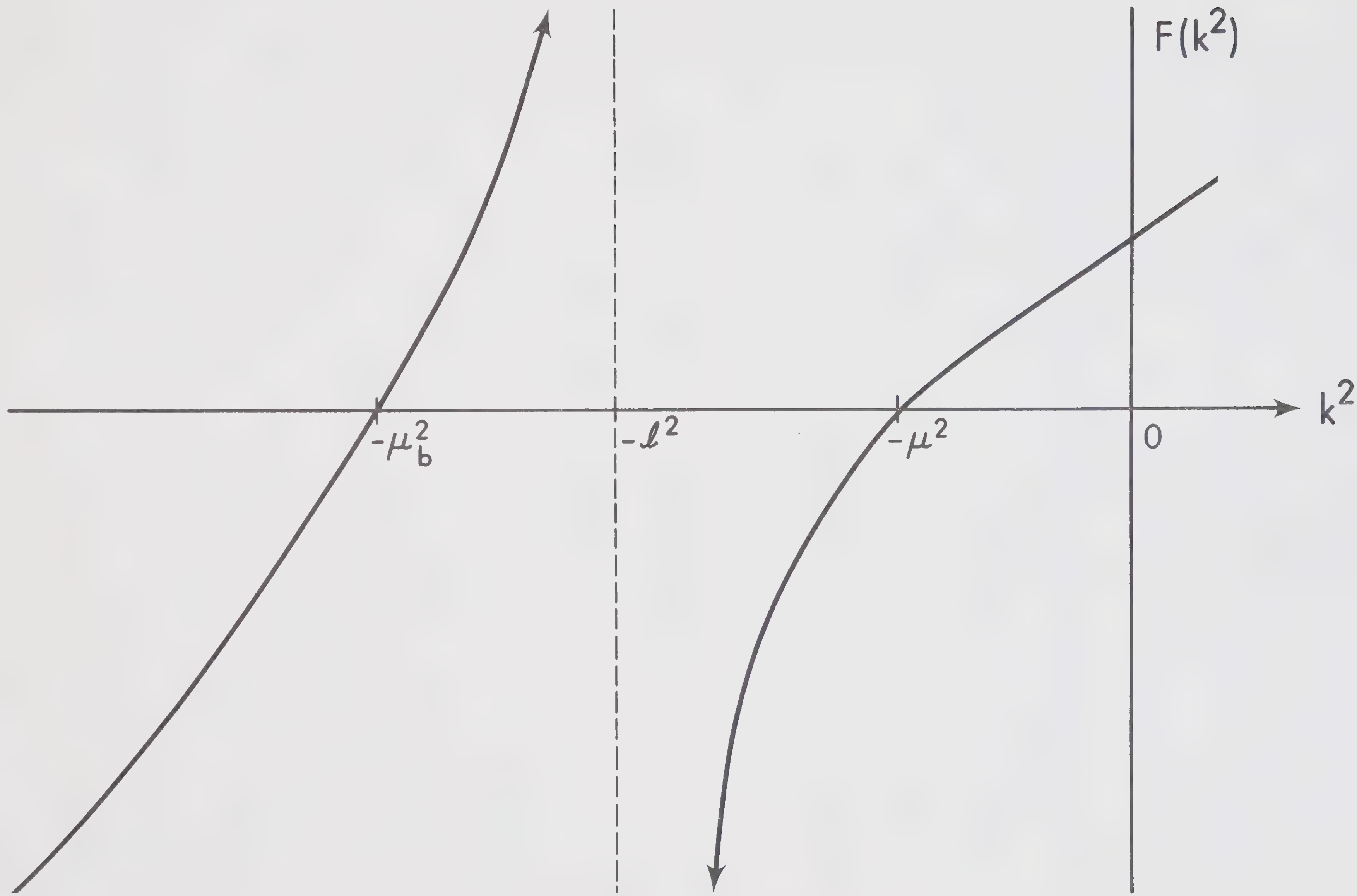
and that $F(k^2)$ has a singular point at $k^2+\ell^2 = 0$ if

$$2g\, J_p(-\ell^2) = 1 \ . \tag{4.95}$$

A sketch graph of $F(k^2)$ for negative (physical) k^2 is given in Figure 3. The behaviour of $F(k^2)$ has been examined numerically to make sure that a solution of (4.95) exists. We have found that it is possible to find combinations of the various parameters, Λ, g and

Figure (3)

A sketch graph of the behaviour of $F(k^2)$ for negative k^2 showing the zero at the bound state mass, μ_b, and at the physical meson mass, μ. The value of ℓ^2 is determined by the solution of equation (4.95).



$F(k^2)$
k^2
$-\mu_b^2$
$-\ell^2$
$-\mu^2$
0

f appearing in our result, that will give a solution for μ_b. The actual numerical value, of course, can be made to vary over a very wide range but it is interesting to notice that for Λ as large as 1000 $4m^2$ we can still make μ_b^2 less than $4m^2$ for not unreasonable values of f and g.

The problem that remains is to determine the signs of the residues of the propagator at $k^2+\mu^2=0$ and at $k^2+\mu_b^2=0$.

If we write

$$F(k^2) = (k^2+\mu^2)(k^2+\mu_b^2)g(k^2) \quad , \tag{4.96}$$

then

$$\begin{aligned}\frac{\partial F}{\partial k^2} &= (k^2+\mu_b^2)g(k^2) + (k^2+\mu^2)g(k^2) \\ &\quad + (k^2+\mu_b^2)(k^2+\mu^2)g'(k^2) \\ &\quad > 0 \quad .\end{aligned} \tag{4.97}$$

Therefore

$$\left.\frac{\partial F}{\partial k^2}\right|_{k^2+\mu^2=0} = (\mu_b^2-\mu^2)g(-\mu^2) > 0$$

or

$$g(-\mu^2) > 0 \tag{4.98}$$

Similarly

$$g(-\mu_b^2) < 0 \quad . \tag{4.99}$$

But using (4.96) we can write the propagator as

$$D_c'(k^2) = -\frac{1}{g(k^2)(\mu_b^2-\mu^2)}\left[\frac{1}{(k^2+\mu^2)} - \frac{1}{(k^2+\mu_b^2)}\right] . \qquad (4.100)$$

It follows that the residues at the two poles have the same sign. There is no ghost.

The situation that we have in this case is just that required by Umezawa* in that we have singularity in $F(k^2)$ between each of its zeros.

It will be remembered that the appearance of the ghost in the Lee Model could be traced to an inconsistency in the definition of the renormalization constant $Z^{(1)}$. We must check that no such inconsistency is built into our theory. The definition of the renormalization constant $Z^{(\mu)}$, that is associated with the state of mass μ, is

$$(Z_\mu)^{-1} = \left.\frac{\partial F(k^2)}{\partial k^2}\right|_{k^2+\mu^2=0} \qquad (4.101)$$

$$= 1 - \frac{f^2 J_p'(-\mu^2)}{(1-2g\, J_p(-\mu^2))^2} . \qquad {}^{\dagger}(4.102)$$

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* See the first section of this chapter.

† $J_p'(-\mu^2) \equiv \left.\dfrac{\partial J_p(k^2)}{\partial k^2}\right|_{k^2+\mu^2=0}$

If we define the physical coupling constant f_{obs} in the usual manner as

$$f_{obs}^2 = Z^{(\mu)}\ f^2 \tag{4.103}$$

we have

$$\frac{1}{f_{obs}^2} = \frac{1}{f^2} - \frac{J_p'(-\mu^2)}{(1-2g\ J_p(-\mu^2))^2} \ . \tag{4.104}$$

In this case the second term on the right hand side is convergent so we have no difficulty. In fact

$$0 < Z^{(\mu)} < 1 \tag{4.105}$$

as it should.

4.4 The Boundstate Coupling Constant

To conclude this chapter we will derive an expression for the coupling constant that governs the fermion, antifermion bound state vertex.

The Bethe-Salpeter amplitude for the fermion field is defined as

$$\chi_q^{\alpha\beta}(x,y) \equiv \langle 0|T(\psi_\alpha(x),\bar{\psi}_\beta(y))|q\rangle \ . \tag{4.106}$$

In Appendix D we use this definition to derive the equations

$$(\gamma\partial(x) + m)\chi_q(x,y)(-\gamma\overleftarrow{\partial}(y) + m)$$

$$= -2ig\ \delta^4(x-y)\gamma_5\ \mathrm{Tr}[\gamma_5\chi_q(x,x)]$$

$$-if\ \delta^4(x-y)(i\gamma_5)\langle 0|\phi(x)|q\rangle \qquad (4.107)$$

and

$$(q^2+\mu^2)\phi_q - \Delta\mu^2\phi_q = -if\ \mathrm{Tr}[\gamma_5\chi_q(0)]\ , \qquad (4.108)$$

where

$$\langle 0|\phi(x)|q\rangle \equiv (2\pi)^{-3/2}\ e^{iqx}\ \phi_q\quad . \qquad (4.109)$$

If we treat the bound state as a stable composite particle we can extend the reduction formalism to include these particles. The main problem in this is to define a suitable interpolating field which tends asymptotically to the required bound state in and out states. There is considerable arbitrariness in any such definition [Lurié, 1968]. A suitable definition that will lead to the required reduction formula is

$$g_B\langle 0|B(x)|q\rangle = [2g + \frac{f^2}{(\mu^2-\mu_b^2-\Delta\mu^2)}]\ \mathrm{Tr}[\gamma_5\chi_q(x,x)]\ , \qquad (4.110)$$

$$q^2+\mu_b^2 = 0\ .$$

In this definition, B(x) is the field that corresponds to the outgoing particle with mass μ_b and g_B is the required coupling constant. This relationship is similar to one used by Aurilia [Aurilia et al, 1972] for a slightly different problem.

The complicated coefficient on the right hand side of (4.110) can be simplified if we remember that the bound state mass was given by

$$\mu^2 - \mu_b^2 - \Delta\mu^2 = \frac{f^2 J_p(-\mu_b^2)}{1 - 2g\, J_p(-\mu_b^2)} \quad . \tag{4.111}$$

Therefore,

$$2g + \frac{f^2}{(\mu^2 - \mu_b^2 - \Delta\mu^2)} = \frac{1}{J_p(-\mu_b^2)} \quad . \tag{4.112}$$

Equation (4.110) can be rewritten as

$$g_B \langle 0|B(x)|q\rangle = \frac{1}{J_p(-\mu_b^2)} (2\pi)^{-2/3} \frac{1}{\sqrt{2q_o}} e^{iqx}$$

$$\times \mathrm{Tr}[\gamma_5 \chi_q(0)] \tag{4.113}$$

after we have factorized out the "centre of mass" dependency.*

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* See Appendix D, equations (D.11)-(D.13).

Formal integration of the B-S equation, (4.107) yields

$$\chi_q(z) = \chi_q^{(o)}(z) + 2g\,\mathrm{Tr}[\gamma_5\chi_q(0)]Q(z;q)$$

$$+ if\,\phi_q\,Q(z;q)\ , \tag{4.114}$$

$$Q(z,q) = -i\int d^4y\ S_c(\tfrac{z}{2}-y)\gamma_5 S_c(\tfrac{z}{2}+y)e^{iqy}\ . \tag{4.115}$$

Use of equations (4.108), (4.109) and (4.114) gives the bound state contribution to the Bethe-Salpeter amplitude as

$$\chi_q^{(B)}(z) = \frac{1}{J_p(-\mu_b^2)}\,Q(z;q)\mathrm{Tr}[\gamma_5\chi_q^{(B)}(0)]\ , \tag{4.116}$$

$$q^2 + \mu_b^2 = 0\ .$$

The normalization condition for the B-S amplitude is*

$$\int d^4z\ d^4z'\ \bar{\chi}_q^{(B)}(z')\ \frac{\partial}{\partial q_o}\ I(z,z',q)\chi_q^{(B)}(z) = \frac{2iq_o}{(2\pi)^4}\ . \tag{4.117}$$

At $q^2+\mu_b^2=0$ this can be expressed in the form

$$|\mathrm{Tr}[\gamma_5\chi_q(0)]|^2 = -J_p(-\mu_b^2)\left(\frac{\partial J_p(q^2)}{\partial q^2}\right)^{-1}_{q^2+\mu_b^2=0} \tag{4.118}$$

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* See, for example, Lurié et al, 1965.

Thus,

$$g_B^2 = -\left(\frac{\partial J_p(q^2)}{\partial q^2}\right)^{-1}_{q^2+\mu_b^2=0} \tag{4.119}$$

or to use the notation we have used previously

$$g_B^2 = -\frac{1}{J_p'(-\mu_b^2)} \quad . \tag{4.120}$$

It is interesting to notice that this result is independent of the coupling constants that appear in the original Lagrangian except for their determination of μ_b.

The model we have discussed in this chapter serves to show that the introduction of a four-fermion interaction leads to the possibility of bound states. What is more important is that these will not necessarily be "ghosts". The bound state we have looked at in this section has fermion number zero but we can easily imagine a bound state of a fermion and a scalar meson that would carry fermion number one. Such a bound state could correspond to an excited state of, say, the electron that we could identify with the muon. It is quite probable that a sufficient mass difference could appear through a $\phi^2(\bar{\psi}\psi)$ interaction that would have many of the feature of the modified Yukawa interaction.

CHAPTER 5

THE PROPAGATOR FOR THE SPIN-ONE FIELD

In Chapter 3 we examined the interaction of a Dirac field with a spin-one field described by a skew-symmetric second rank tensor. It was shown, that when the gauge is fixed in such a way as to regain the usual Proca equation, the interaction reduced to the form given by minimal coupling except that in this case we also have a four-fermion term. It was speculated that this latter term could produce fermion-antifermion bound states in the same way as a four-fermion interaction does in the modified Yukawa case.

In this chapter we will investigate this possibility in more detail using the same method we used for the pseudo-scalar meson theory.

5.1 The Propagator

The equations we arrived at in Chapter 3 when the gauge was fixed were:

$$(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu - \mu_o^2\delta_{\mu\nu})V_\nu(x) = J_\mu(x) \quad ; \tag{5.1}$$

$$J_\mu(x) = -\, ig\,\bar{B}(x)\gamma_\mu B(x) \quad ; \tag{5.2}$$

$$-(\gamma_\mu \partial_\mu + m_o) B(x) = -igV_\rho(x)\gamma_\rho B(x)$$

$$+ \frac{ig^2}{\mu_o^2} (\bar{B}(x)\gamma_\rho B(x)) i\gamma_\rho B(x) \quad . \tag{5.3}$$

The subscript zero denotes the fact that the mass parameters that appear are the bare masses.

The principal strength of our formulation is that we have a definite prescription for limiting the possible form of the interaction Lagrangian. This has led us to a theory which depends on only one coupling constant, g. This constant was related to the constant G that appeared in our original Lagrangian (3.8)-(3.9) by

$$g = \sqrt{2}\, \mu_o\, G \quad . \tag{5.4}$$

This is a rather unusual relationship: The bare coupling constant depends on the bare mass! (The coupling constant for the four-fermion part of the interaction, however, is g^2/μ_o^2 which is independent of μ_o.)

In the usual way, we can write the equations of motion (5.1)-(5.3) in terms of the observed masses, μ_{obs} and m_{obs} of the meson and fermion particle respectively:

$$(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu - \mu_{obs}^2\delta_{\mu\nu})V_\nu = J_\mu - \Delta\mu^2 V_\mu(x) \quad ; \tag{5.5}$$

$$-(\gamma_\mu \partial_\mu + m_{obs}) B(x) = -ig\, V_\rho \gamma_\rho B(x)$$

$$+ \frac{ig^2}{\mu_o^2} (\bar{B}\gamma_\rho B) i\gamma_\rho B(x) - \delta m\, B(x) \quad , \tag{5.6}$$

where

$$\mu_{obs}^2 - \Delta\mu^2 \equiv \mu_o^2 \quad . \tag{5.7}$$

Following the method we used in the previous chapter, we define the unrenormalized propagator, $D_c'(x-x')$, as

$$iD_{c\mu\nu}'(x-x') \equiv \langle 0|T^*(V_\mu(x),V_\nu(x'))|0\rangle \tag{5.8}$$

where

$$T^*(V_\mu(x),V_\nu(x')) = T(V_\mu(x),V_\nu(x'))$$

$$- \frac{i}{2}\,[\varepsilon(x_o-x_o'),d_{\mu\nu}(\partial)]\Delta(x-x') \quad . \tag{5.9}$$

If we make use of the equation of motion (5.5) and the result that the Schwinger terms that appear in the various equal time commutation relationships are of the form derivative of the four delta-function [Roman, 1969], then we find that

$$i(\Box\delta_{\mu\alpha} - \partial_\mu\partial_\alpha - \mu_{obs}^2\delta_{\mu\alpha})D_{\alpha\nu}'(x-x')$$

$$= \langle 0|T(\eta_\mu(x),V_\nu(x'))|0\rangle + i\delta^4(x-x')\delta_{\mu\nu}, \tag{5.10}$$

$$\eta_\mu(x) = -i\,g\,\bar{B}\gamma_\mu B - \Delta\mu^2 V_\mu(x) \quad . \tag{5.11}$$

Let us define a vertex function, $\Gamma_\mu(x-y)$, by

$$<0|T(V_\mu(x), B(y), \bar{B}(z))|0>$$

$$\equiv ig \int_{-\infty}^{\infty} d^4x' \; d^4y' \; d^4z' \; S_c'(y-y')\Gamma_\nu(y'-x';x'-z')$$

$$\times S_c'(z'-z) D_{\nu\mu}'(x-x') \quad . \tag{5.12}$$

Therefore,

$$<0|T(V_\mu(x), \bar{B}(y)\gamma_\rho B(y))|0>$$

$$= - \frac{ig}{(2\pi)^8} \int_{-\infty}^{\beta} d^4p \; d^4q \; \mathrm{Tr}[\gamma_\rho S_c'(p)\Gamma_\nu(p;p-q)$$

$$\times S_c'(p-q)] \; D_{\nu\mu}'(q) \; e^{iq(y-x)} \quad . \tag{5.13}$$

To reach this result we have introduced the Fourier transforms of the functions $D_{\mu\nu}'(x-x')$, $S_c'(x-x')$ and $\Gamma_\nu(x;y)$. For example, the Fourier transform of the causal Dirac propagator, $S_c'(x-x')$, is defined as

$$S_c'(x-x') \equiv \frac{1}{(2\pi)^4} \int d^4p \; e^{ip(x-x')} S_c'(p) \quad . \tag{5.14}$$

We can now introduce the quantity $\Pi^*(q)$ which corresponds, in some way, to the renormalized vector-vector closed loop diagram:

$$\Pi_{\mu\nu}^*(q) \equiv - \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} d^4p \; \mathrm{Tr}[\gamma_\mu S_c'(p)\Gamma_\nu(p;p-q)S_c'(p-q)] \quad . \tag{5.15}$$

Using this definition, equation (5.13) can be rewritten

in the form

$$<0|T(V_\mu(x),\bar{B}(y)\gamma_\rho B(y))|0>$$

$$= \frac{g}{(2\pi)^4} \int_{-\infty}^{\infty} d^4q \; \Pi^*_{\rho\nu}(q) D'_{\nu\mu}(q) \; e^{iq(x-y)} \; . \tag{5.16}$$

The substitution of equation (5.16) into (5.10) yields

$$i(\Box\delta_{\mu\alpha} - \partial_\mu\partial_\alpha - \mu^2_{obs}\delta_{\mu\alpha})D'_{\alpha\nu}(x-x')$$

$$= -\frac{ig^2}{(2\pi)^4}\int d^4q \; e^{iq(x-x')} \; \Pi^*_{\mu\alpha}D'_{\alpha\nu}(q)$$

$$- \frac{i\Delta\mu^2}{(2\pi)^4}\int d^4q \; e^{iq(x-x')} \; D'_{\mu\nu}(q)$$

$$+ \frac{i}{(2\pi)^4}\int d^4q \; e^{iq(x-x')} \; \delta_{\mu\nu} \; , \tag{5.17}$$

or

$$(q^2\delta_{\mu\alpha} - q_\mu q_\alpha + \mu^2_{obs}\delta_{\mu\alpha})D'_{\alpha\nu}(q)$$

$$= g^2\Pi^*_{\mu\alpha}D'_{\alpha\nu}(q) + \Delta\mu^2 D'_{\mu\nu}(q) - \delta_{\mu\nu} \; . \tag{5.18}$$

The task that remains is to find an expression for the vector-vector closed loop integral $\Pi^*_{\mu\nu}(q)$. To do this we again employ the pair-approximation. In this approximation we can expand the closed loop diagram as in Figure 2, remembering, of course, that now we are dealing with vector-vector coupling. Thus,

$$\Pi^*_{\mu\nu}(q) = -\frac{i}{(2\pi)^4}\int d^4p\ \mathrm{Tr}[\gamma_\mu S_c(p)\Gamma_\nu(p;p-q)S_c(p-q)] \tag{5.19}$$

$$= \frac{i}{(2\pi)^4}\int d^4p\ \mathrm{Tr}[\gamma_\mu S_F(p)\Gamma_\nu(p;p-q)S_F(p-q)]$$

$$= J_{\mu\nu} + (-f)J_{\mu\alpha}J_{\alpha\nu} + (-f)^2 J_{\mu\alpha}J_{\alpha\beta}J_{\beta\nu} + \ldots\ldots\ldots \quad , \tag{5.20}$$

where

$$f = \frac{g^2}{\mu_o^2} = 2G \quad , \tag{5.21}$$

and

$$J_{\mu\nu}(q) = \frac{i}{(2\pi)^4}\int d^4p\ \mathrm{Tr}[\gamma_\mu S_F(p)\gamma_\nu S_F(p-q)] \ . \tag{5.22}$$

As in the case of the pseudoscalar-pseudoscalar closed loop integral $J_p(q^2)$ it is much more convenient to deal with the dispersive form of (5.22), i.e.

$$J_{\mu\nu}(q) = J^V(q^2)(q_\mu q_\nu - q^2\delta_{\mu\nu}) + J^S(q^2)\delta_{\mu\nu} \ , \tag{5.23}$$

$$J^V(q^2) = \int_{4m^2}^{\infty} dx^2\ \frac{\rho(x^2)}{(q^2 + x^2 - i\varepsilon)} \quad , \tag{5.24}$$

$$J^S(q^2) = \int_{4m^2}^{\infty} dx^2\ \frac{x^2\rho(x^2)}{(q^2 + x^2 - i\varepsilon)} \quad , \tag{5.25}$$

and

$$\rho(x^2) = \frac{1}{12\pi^2}\left(1 - \frac{4m^2}{x^2}\right)^{\frac{1}{2}}\left(1 + \frac{2m^2}{x^2}\right) \qquad {}^{*}(5.26)$$

The sum on the right hand side of equation (5.20) can be expressed as

$$(q^2\delta_{\mu\nu} - q_\mu q_\nu)(A + (-f)A^2 + (-f)^2A^3 + \ldots.)$$

$$+ (q_\mu q_\nu)(B + (-f)B^2 + (-f)^2B^3 + \ldots.) \ , \qquad (5.27)$$

where for convenience we have taken the combinations

$$\frac{J_S(q^2)}{q^2} - J^V(q^2) \equiv A \qquad (5.28)$$

and

$$J^V(q^2) = B \ . \qquad (5.29)$$

Hence

$$\Pi^{*}_{\mu\nu}(q) = \frac{(q^2\delta_{\mu\nu} - q_\mu q_\nu)A}{(1 + fq^2A)} + \frac{q_\mu q_\nu B}{(1 + fq^2B)} \ . \qquad (5.30)$$

Returning to equation (5.18) we obtain

$$D'_{\mu\nu}(q) = D_1(q^2)(q^2\delta_{\mu\nu} - q_\mu q_\nu) + D_2(q^2)q_\mu q_\nu \ , \qquad (5.31)$$

where

$$D_1(q^2) = \frac{-1}{q^2[q^2+\mu_o^2 - g^2q^2(\frac{A}{1 + fq^2A})]} \qquad (5.32)$$

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* Equations (5.23)-(5.26) are derived in Appendix C.

and

$$D_2(q^2) = \frac{-1}{q^2[\mu_o^2 - g^2q^2(\frac{B}{1 + fq^2B})]} \quad . \tag{5.33}$$

We know, however, that as far as S-matrix calculations are concerned we can ignore the terms in the propagator proportional to $q_\mu q_\nu$. This follows in exactly the same way as in Quantum Electrodynamics so long as the vector field is coupled to a conserved current. Effectively we have

$$D'_{\mu\nu}(q) = - \frac{\delta_{\mu\nu}}{q^2 + \mu_o^2 - g^2q^2(\frac{A}{1 + fq^2A})} \quad . \tag{5.34}$$

The mass correction, $\Delta\mu^2$, is determined by the condition that the propagator should have a pole at the physical meson mass μ. This demands that

$$- \Delta\mu^2 + g^2\mu^2 \quad \frac{A(-\mu^2)}{(1 - f\mu^2A(-\mu^2))} = 0 \tag{5.35}$$

or

$$\Delta\mu^2 = \frac{g^2\mu^2A(-\mu^2)}{(1 - f^2\mu^2A(-\mu^2))} \quad . \tag{5.36}$$

Using the definition of g, (5.4), we can simplify this result:

$$\Delta\mu^2 = f\mu^4A(-\mu^2) \ . \tag{5.37}$$

Before proceeding to see if there is another pole in the propagator we have obtained corresponding to a

bound state, we should check to see that the results we have are in agreement with the Bethe-Salpeter method.

The Bethe-Salpeter amplitude for the fermion field is defined as

$$\chi_q^{\alpha\beta}(x,y) = \langle 0|T(B_\alpha(x),\bar{B}_\beta(y))|q\rangle \quad . \tag{5.38}$$

Applying the equations of motion we get

$$\begin{aligned}(\gamma\vec{\partial}_{(x)} + m)_{\alpha\alpha'}\, \chi_q^{\alpha'\beta'}(x,y)(-\gamma\overleftarrow{\partial}_{(y)} + m)_{\beta'\beta} \\ = -i\delta(x_o - y_o)\gamma_4^{\beta'\beta}\langle 0|\{j_\alpha(x),\bar{B}_{\beta'}(y)\}|q\rangle \\ \langle 0|T(j_\alpha(x),\bar{j}_\beta(y))|0\rangle \quad .\end{aligned} \tag{5.39}$$

The function j(x) is just the source of the Dirac field, i.e.

$$j_\alpha(x) = igV_\rho(x)\gamma_\rho B_\alpha(x) + f(\bar{B}\gamma_\rho B)\gamma_\rho B^\alpha(x) + \delta m\, B_\alpha(x) \quad . \tag{5.40}$$

In the pair-approximation we can ignore the second term on the right hand side of equation (5.39) so

$$\begin{aligned}(\gamma\vec{\partial}_{(x)} + m)_{\alpha\alpha'}\chi^{\alpha'\beta'}(x,y)(-\gamma\overleftarrow{\partial}_{(y)} + m)_{\beta'\beta} \\ = g(\gamma_\rho)^{\alpha\beta}\ \delta^4(x-y)\langle 0|V_\rho|q\rangle \\ + if\ \delta^4(x-y)(\gamma_\rho)^{\alpha\beta}\ \mathrm{Tr}[\gamma_\rho\chi_q(x,x)] \quad .\end{aligned} \tag{5.41}$$

This equation can be formally integrated to give

$$\chi_q(x,y) = \chi_q^o(x,y) + g\int d^4u \; S_c(x-u)\gamma_\rho S_c(x-u)<0|V_\rho|q>$$

$$+ if\int d^4u \; S_c(x-u)\gamma_\rho S_c(u-y)Tr[\gamma_\rho\chi_q(x,x)] \; . \qquad (5.42)$$

If we define

$$<0|V_\rho|q> = (2\pi)^{-3/2} \; e^{iqx} \; V_\rho(q) \qquad (5.43)$$

and

$$Q_\rho(z;q) = -i\int d^4y \; S_c(\tfrac{z}{2} - y)\gamma_\rho S_c(\tfrac{z}{2} + y) \; e^{iqy} \quad . \qquad (5.44)$$

We can rewrite (5.42) in the form

$$\chi_q(z) = \chi_q^{(o)}(z) + ig \; Q_\rho(z;q)V_\rho(q)$$

$$- f \; Q_\rho(z;q) \; Tr[\gamma_\rho\chi_q(0)] \; . \qquad (5.45)$$

To reach this result we have factorised out the "centre of mass" dependency in the usual way.*

If we multiply this equation by γ_ν and take the trace of both sides putting

$$Tr[\gamma_\nu\chi_q(0)] = C_\nu(q) \quad , \qquad (5.46)$$

we obtain

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* See Appendix D, equations (D.11)-(D.13).

$$C_\nu(q) = C^o_\nu(q) + igJ_{\nu\rho}(q)V_\rho(q) - fJ_{\nu\rho}(q)C_\rho(q) \quad . \qquad (5.47)$$

In this equation we have reintroduced the notation we have used previously for the fermion closed loop integral, i.e. $J_{\nu\rho}(q)$. Equation (5.47) and that that comes from the equation of motion for vector field in this notation,

$$(q^2 + \mu_o^2)V_\mu(q) = -igC_\mu(q) \quad , \qquad (5.48)$$

can be used to determine the mass correction $\Delta\mu^2$. We first set $C^o_\nu(q)$ equal to zero and $q^2+\mu^2=0$ and look for a solution of the simultaneous equations (5.48) and (5.47). We find that

$$\Delta\mu^2 = f\mu^4 A(-\mu^2) \quad . \qquad (5.49)$$

The function $A(q^2)$ is just that introduced by equation (5.28). Comparison of equations (5.49) and (5.37) shows that the two methods give the same result for $\Delta\mu^2$.

5.2 Conclusion

Now that we have determined the form of the propagator for the vector field we are in a position to see whether or not there is a bound state in our theory.

Equation (5.34) says that

$$D'_{\mu\nu}(q) = - \frac{\delta_{\mu\nu}}{q^2 + \mu_o^2 - g^2 q^2 \left(\frac{A(q^2)}{1+ fq^2 A(q^2)}\right)} \tag{5.50}$$

which we can write as

$$D'_{\mu\nu}(q) = - \frac{\delta_{\mu\nu}}{F(q^2)} \quad . \tag{5.51}$$

A bound state corresponds to a second zero in the function $F(q^2)$ at, say, $q^2+\mu_b^2=0$. ($F(q^2)$ has a zero at $q^2+\mu^2= 0$ by construction.) We can use the result we have obtained for the mass correction to give

$$F(q^2) = q^2 + \mu^2 - \frac{g^2\mu^2(A(-\mu^2))}{(1- f\mu^2 A(-\mu^2))} - \frac{g^2 q^2 A(q^2)}{(1+ fq^2 A(q^2))}$$

or

$$F(q^2) = q^2 + \mu^2 - g^2 \left(\frac{\mu^2 A(-\mu^2) + q^2 A(q^2)}{(1- f\mu^2 A(-\mu^2))(1+ fq^2 A(q^2))}\right) . \tag{5.52}$$

If we use the definition of $A(q^2)$ given by (5.28) we can see that

$$\frac{\partial F(q^2)}{\partial q^2} > 0 \ , \tag{5.53}$$

so for a bound state to exist $F(q^2)$ must have a similarity at, say, $q^2+ \ell^2= 0$, where

$$\mu_b^2 > \ell^2 > \mu^2 \tag{5.54}$$

and

$$(1 - f\ell^2 A(-\ell^2)) = 0 \ . \tag{5.55}$$

These conditions have the same form as those we obtained for the modified Yukawa interaction except that now we only have one free parameter, f. In fact equation (5.55) and the equation for the bound state mass μ_b,

$$F(-\mu_b^2) = 0 \tag{5.56}$$

give us two equations for f in terms of ℓ, the singularity point and μ_b. For any value of the cut-off that must be introduced into the integrals that appear in the definition of $A(q^2)$, these two equations are inconsistent with the restriction that ℓ lies between μ and μ_b. We are therefore led to the conclusion that no bound state exists in this theory within our approximation.

APPENDIX A

NOTATION AND CONVENTIONS

The following notation and conventions are used throughout this thesis.

1) Natural units are employed in which

$$\hbar = c = 1$$

2) Summation over repeated indices is implicitly assumed unless the contrary is stated.

3) The fourth component of a four-vector is imaginary. For example

$$x_\mu = (\underset{\sim}{x}, ix_o) .$$

No distinction is made between contravariant and covariant vectors.

4) The volume elements which occur in integrals are denoted by either

$$d^4x = dx_o\, dx_1\, dx_2\, dx_3$$

or $$d^3x = dx_1\, dx_2\, dx_3 .$$

5) For the four-gradient differential operator operating to the right we use

$$\partial_\mu \equiv \frac{\partial}{\partial x_\mu} = (\underset{\sim}{\nabla}, -i\,\frac{\partial}{\partial x_o}),$$

and similarly

$$\partial_\mu' \equiv \frac{\partial}{\partial x_\mu'} = (\underset{\sim}{\nabla}', -i\,\frac{\partial}{\partial x_o'}) .$$

6) The differential operator $\overleftarrow{\partial}_\mu$ operates on functions standing to its left. For example if f(x) and g(x) are arbitrary functions

$$f(x)\overleftarrow{\partial}_\mu g(x) = (\partial_\mu f(x))g(x)$$

and

$$f(x)(\partial_\mu + \overleftarrow{\partial}_\mu)g(x) = \partial_\mu(f(x)g(x)) .$$

7) The D'Alembertian operator $\Box$ is

$$\Box = \partial_\mu\partial_\mu = \nabla^2 - \frac{\partial^2}{\partial x_o^2} .$$

8) The Klein-Gordon divisor $d(\partial)$, corresponding to the differential operator $\Lambda(\partial)$, is defined by the relationship

$$\Lambda(\partial)d(\partial) = (\Box - m^2) .$$

9) The matrix tensor $g_{\mu\nu}$ is such that

$$\begin{aligned} g_{\mu\nu} &= 1 && \text{for} \quad \mu=\nu=1,2,3, \\ &= -1 && \text{for} \quad \mu=\nu=4, \\ &= 0 && \text{otherwise} . \end{aligned}$$

10) The Levi-Civita symbol, $\varepsilon_{\mu\nu\sigma\rho}$, is defined as follows:

$$\begin{aligned} \varepsilon_{\mu\nu\sigma\rho} &= 1 \text{ if } (\mu\nu\sigma\rho) \text{ is an even permutation of } (1234) \\ &= -1 \text{ if } (\mu\nu\sigma\rho) \text{ is an odd permutation of } (1234) \\ &= 0 \text{ otherwise.} \end{aligned}$$

Hence,

$$\varepsilon_{\mu\nu\sigma\rho}\delta_{\mu\nu} = 0$$

$$\varepsilon_{\mu\nu\sigma\rho} = - \varepsilon_{\mu\nu\sigma\rho} \quad .$$

Also we have the relationship:

$$\varepsilon_{\mu\nu\sigma\rho}\varepsilon_{\mu\nu\sigma\rho} = 4!$$

$$\varepsilon_{\mu\nu\sigma\rho}\varepsilon_{\alpha\nu\sigma\rho} = 3!\ \delta_{\alpha\mu}$$

$$\varepsilon_{\mu\nu\sigma\rho}\varepsilon_{\alpha\beta\sigma\rho} = 2!\ (\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})$$

$$\begin{aligned}\varepsilon_{\mu\nu\sigma\rho}\varepsilon_{\alpha\beta\gamma\rho} = 1!\ (&\delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\sigma\gamma} - \delta_{\mu\alpha}\delta_{\nu\gamma}\delta_{\sigma\beta} \\ &+ \delta_{\mu\beta}\delta_{\nu\gamma}\delta_{\sigma\alpha} - \delta_{\mu\beta}\delta_{\nu\alpha}\delta_{\sigma\gamma} \\ &+ \delta_{\mu\gamma}\delta_{\nu\alpha}\delta_{\sigma\beta} - \delta_{\mu\gamma}\delta_{\nu\beta}\delta_{\sigma\alpha}) \ .\end{aligned}$$

11) A time-like unit vector is denoted by n_μ,

$$n_\mu n_\mu = -1 \ .$$

12) A space-like surface in four dimensional Minkowski space is denoted by σ, or if we wish to specify that the surface passes through the point x, by $\sigma(x)$. The four-dimensional surface area $d\sigma_\mu(x)$ is defined as

$$d\sigma_\mu(x) = (dx_2dx_3dx_o,\ dx_1dx_3dx_o,\ dx_1dx_2dx_o, -i\ dx_1dx_2dx_3) \ .$$

13) The usual symbols for commutation and anticommutation relationships are used, i.e.

$$[A,B] = AB - BA$$

$$\{A,B\} = AB + BA$$

14) The symbol † is reserved for hermitian conjugation.

15) Field quantities in the Heisenberg picture are represented, for example, by $\underset{\sim}{\psi}_{\mu\nu}(x)$ while those in the interaction picture are represented by $\psi_{\mu\nu}(x)$. This convention is carried over to the case of currents, hence

$$\underset{\sim}{J}_{\mu} = -i\underset{\sim}{\bar{B}}(x)\gamma_{\mu}\underset{\sim}{B}(x) \ ,$$

while

$$J_{\mu} = -i\bar{B}(x)\gamma_{\mu}B(x) \ .$$

16) The T*-product is defined by

$$<0|T^*(\phi_\alpha(x),\bar{\phi}_\beta(x'))|0> = id_{\alpha\beta}(\partial)\Delta_c(x-x') \ .$$

It is related to the "Wick's chronological" product, T, by

$$T(\phi_\alpha(x),\phi_\beta(x')) = T^*(\phi_\alpha(x),\phi_\beta(x')) + \frac{i}{2}[\varepsilon(x_o-x_o')d_{\alpha\beta}(\partial)]\Delta(x-x').$$

APPENDIX B

SOLUTIONS OF THE KLEIN-GORDON EQUATION

The following functions related to the solution of the Klein-Gordon equation are defined as:

$$\Delta(x) = i\,(2\pi)^{-3} \int_{-\infty}^{\infty} d^4k\ \varepsilon(k_o)\ \delta(k^2+m^2)\ e^{ikx}\ ;$$

$$\Delta^{(1)}(x) = (2\pi)^{-3} \int_{-\infty}^{\infty} d^4k\ \delta(k^2+m^2)\ e^{ikx}\ ;$$

$$\Delta^{(\pm)}(x) = \mp i\,(2\pi)^{-3} \int_{-\infty}^{\infty} d^4k\ \theta(\pm k_o)\ \delta(k^2+m^2)\ e^{ikx}\ .$$

They are related by

$$\Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x)\ ,$$

$$\Delta^{(1)}(x) = i\Delta^{(+)}(x) - i\Delta^{(-)}(x)\ ,$$

$$\Delta^{(\pm)}(x) = \frac{1}{2}\,(\Delta(x) \mp i\Delta^{(1)}(x))\ .$$

It is evident that

$$(\Box - m^2)\Delta(x) = (\Box - m^2)\Delta^{(1)}(x) = (\Box - m^2)\Delta^{(\pm)}(x) = 0\ ,$$

and

$$\Delta(x) = -\,\Delta(-x)$$

$$\Delta(x) = -\,\Delta(x^*)$$

$$\Delta^{(+)}(x) = -\Delta^{(-)}(-x)$$

$$\Delta(x, \zeta_o=0) = 0\ .$$

Also,

$$\delta(x_o)\partial_o\Delta(x) = -\delta^4(x)$$

$$\delta(x_o)\partial_o\Delta^{(+)}(x) = -\frac{1}{2}\,\delta^4(x)$$

$$\delta(x_o)\partial_o\Delta^{(-)}(x) = -\frac{1}{2}\,\delta^4(x)$$

$$\delta(x_o)\partial_o\Delta^{(1)}(x) = 0\ .$$

We further use the functions

$$\Delta^{ret}(x) = \theta(x_o)\Delta(x) \quad ,$$

$$\Delta^{adv}(x) = \theta(-x_o)\Delta(x) \quad ,$$

$$\Delta_c(x) = \theta(x_o)\Delta^{(+)}(x) - \theta(-x_o)\Delta^{(-)}(x)$$

$$= -\frac{1}{(2\pi)^4}\int d^4k\,\frac{1}{k^2+m^2-i\varepsilon}\quad e^{ikx}\ ,$$

where

$$(\Box - m^2)\Delta^{adv}(x) = -\delta^4(x)$$

$$(\Box - m^2)\Delta^{ret}(x) = (\Box - m^2)\Delta_c(x) = \delta^4(x)\ .$$

APPENDIX C

THE DISPERSIVE FORMS OF THE FERMION CLOSED LOOP INTEGRALS

The pair-approximation applied to the modified Yukawa interaction considered in Chapter 4 necessitates the evaluation of the integral $J_p(k^2)$ corresponding to a pseudoscalar-pseudoscalar fermion closed loop. In this appendix we will show how the dispersive form of this integral is derived. The general method we shall describe for $J_p(k^2)$ is quite cumbersome to carry out in practice so we will also list a number of very useful formulae for these recurring types of integrals. These formulae are applied to the vector-vector integral, $J_{\mu\nu}(k)$, to derive the dispersive form we used in Chapter 5.

Equation (4.87) defines $J_p(k^2)$ as

$$J_p(k^2) = \frac{i}{(2\pi)^4} \int d^4p \; \mathrm{Tr}[\gamma_5 S_F(p) \gamma_5 S_F(p-k)] \tag{C.1}$$

where

$$S_F(x) = \frac{1}{(2\pi)^4} \int d^4p \; e^{ipx} \; S_F(p) \tag{C.2}$$

$$S_F(p) = - \frac{1}{(\gamma.p - im - i\varepsilon)} \quad .$$

As we have shown, it is much more convenient for practical purposes to rewrite the expression in (C.1) in the form of a dispersive integral. Relativistic

invariance requires that $J_p(k^2)$ be a function of k^2 only. This can be seen by applying the standard Feynman techniques. Using (C.3), we can write (C.1) as

$$J_p(k^2) = \frac{i}{(2\pi)^4} \int d^4p \, \mathrm{Tr}[\gamma_5 \frac{1}{(\gamma.p-im)} \gamma_5 \frac{1}{(\gamma.(p-k)-im)}] \tag{C.4}$$

$$= - \frac{4i}{(2\pi)^4} \int d^4p \, \frac{p^2+ m^2- p.k}{(p^2+m^2)[(p-k)^2+m^2]} \quad . \tag{C.5}$$

By using the Feynman formula

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2} \tag{C.6}$$

we can cast the denominator in (C.5) into the form

$$\frac{1}{(p^2 + m^2)((p-k)^2+ m^2)}$$

$$= \int_0^1 dz \, \frac{1}{\{(p^2+m^2)z+[(p-k)^2+m^2](1-z)\}^2}$$

$$= \int_0^1 dz \, \frac{1}{[p^2+m^2+(k^2-2(p.k))(1-z)]^2}$$

$$= \int_0^1 dz \, \frac{1}{\{[p-k(1-z)]^2+m^2+k^2(1-z)z\}^2} \quad . \tag{C.7}$$

If we make the change of variable

$$p \to p + k(1 - z) \tag{C.8}$$

we can remove the terms linear in k and arrive at the expression

$$J_p(k^2) = -\frac{4i}{(2\pi)^4}\int_0^1 dz\, d^4p\, \frac{p^2+m^2-k^2(1-z)z+(p.q)(1-2z)}{[p^2+m^2+q^2(1-z)z]^2} \tag{C.9}$$

The denominator is entirely in terms of p^2 so we can drop the linear term in p from the numerator as it will not contribute to the integral. Now the whole integral only depends on q^2.

The next step in writing $J_p(k^2)$ in dispersive form is to transform the d^4p integration in (C.9) into an integration over a 4-dimensional euclidean space by making the change of variable

$$p_o \to ip_o \tag{C.10}$$

so that

$$d^4p \to id^4p$$

and

$$p^2 - p_o^2 \to p^2 + p_o^2 \quad . \tag{C.11}$$

This charge of variable corresponds to a Wick rotation of 90° which is justified so long as we do not make the

contour of integration cross any singularities. It is easily seen from (C.2) and (C.3) that the rotation is justified in this case.

Since the integral depends only on p^2 we may introduce polar coordinates so that the $\int d^4p$ integration is just

$$\int d^4p \rightarrow 2\pi^2 \int |p|^3 \ d|p| \quad . \tag{C.12}$$

It is then a simple matter in principle to perform the z integration, whence we arrive at the result

$$J_p(k^2) = \int_{4m^2}^{\infty} \frac{\rho(x^2)\ dx^2}{(k^2 + x^2)} \quad , \tag{C.13}$$

where the spectral function $\rho(x^2)$ is given by

$$\rho(x^2) = \frac{1}{8\pi^2}\ x^2(1 - \frac{4m^2}{x^2}) \quad . \tag{C.14}$$

Rather than apply this direct method to evaluate the dispersive form of $J_{\mu\nu}(k)$, (5.23)-(5.26), we will list several useful formulae for handling integrals of this type.

(1) $$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}$$

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^1 dy \frac{x}{[A(1-x) + Bxy + Cx(1-y)]^3}$$

$$\frac{1}{A^n} - \frac{1}{B^n} = -\int_0^1 dz \frac{n(A-B)}{[(A-B)z + B]^{n+1}}$$

(2) $$\frac{1}{(x+a)} = \int_x^\infty dy \frac{1}{(y+a)^2}$$

$$\frac{1}{(x+a)^2} = 2\int_x^\infty dy \frac{1}{(y+a)^3} = 3!\int_x^\infty dy \int_y^\infty dz \frac{1}{(z+a)^4}$$

(3) $$\int_{-\infty}^{\infty} d^4p \frac{(p^2)^{m-2}}{[p^2+\Lambda-i\varepsilon]^n} = \frac{i\pi^2}{(\Lambda-i\varepsilon)^{n-m}} \frac{(m-1)!(n-m-1)!}{n!} \qquad n>m>0$$

$$\int_{-\infty}^{\infty} d^4p \frac{1}{[p^2+\Lambda-i\varepsilon]^3} = \frac{i\pi^2}{\Lambda-i\varepsilon} \frac{1}{2}$$

$$\int_{-\infty}^{\infty} d^4p \frac{1}{[p^2+\Lambda-i\varepsilon]^4} = \frac{i\pi^2}{(\Lambda-i\varepsilon)^2} \frac{1}{6}$$

$$\int_{-\infty}^{\infty} d^4p \frac{p^2}{[p^2+\Lambda-i\varepsilon]^n} = \frac{i\pi^2}{(\Lambda-i\varepsilon)^{n-2}} \frac{1}{(n-1)(n-2)} \qquad (n>2)$$

$$\int_{-\infty}^{\infty} d^4p \frac{p^2}{[p^2+\Lambda-i\varepsilon]^4} = \frac{i\pi}{\Lambda-i\varepsilon} \frac{1}{3}$$

$$\int_{-\infty}^{\infty} d^4p\ e^{-ip^2x} = -\frac{i\pi^2}{x} \qquad (x > 0)$$

(4) $$\int_0^1 dx\ \theta(1-x)\kappa^2 - m^2) = 1 - \frac{m^2}{\kappa^2}$$

$$\int_0^1 dx\ (1-x)\theta((1-x)\kappa^2 - m^2) = \frac{1}{2}\ (1 - \frac{m^4}{\kappa^4})$$

$$\int_0^1 dx\,x\ \theta((1-x)\kappa^2 - m^2) = \frac{1}{2}\ (1 - \frac{m^2}{\kappa^2})^2$$

$$\int_0^1 \frac{dx}{A + Bx} = \frac{1}{B}\log(1 + \frac{B}{A})$$

$$\int_0^1 \frac{x\ dx}{A + Bx} = \frac{1}{B}\ [1 - \frac{A}{B}\log(1 + \frac{B}{A})]$$

(5) $$\int_0^1 dx \int_{m^2}^{\infty} dM^2\ \delta(M^2 - \frac{1}{4}(1 - x^2)\kappa^2) = \int_0^1 dx\ \theta(\frac{1}{4}(1 - x^2)\kappa^2 - m^2)$$

$$= \int_0^{(1 - \frac{4m^2}{\kappa^2})^{\frac{1}{2}}} dx = (1 - \frac{4m^2}{\kappa^2})^{\frac{1}{2}}$$

$$\int_0^1 dx \int_{m^2}^{\infty} dM^2 \int_{M^2}^{\infty} d\mu^2 \; \delta(\mu^2 - \tfrac{1}{4}(1 - x^2)\kappa^2)$$

$$= \int_0^1 dx \int_{m^2}^{\infty} dM^2 \; \theta(\tfrac{1}{4}(1 - x^2)\kappa^2 - M^2)$$

$$= \int_0^1 dx \int_0^{\infty} dM^2 \; \theta(M^2 - m^2) \; \theta(\tfrac{1}{4}(1 - x^2)\kappa^2 - M^2)$$

$$= \frac{\kappa^2}{6} \left(1 - \frac{4m^2}{\kappa^2}\right)^{3/2}$$

If we apply these formulae we can obtain the following results for the fermion closed loop integrals:

$$S_c(p) \equiv - \frac{i\gamma p - m}{p^2 + m^2 - i\varepsilon}$$

$$J^P(q^2) \equiv \int_{4m^2}^{\infty} d\kappa^2 \frac{\kappa^2}{q^2 + \kappa^2 - i\varepsilon} \frac{1}{8\pi^2} [1 - \frac{4m^2}{\kappa^2}]^{\frac{1}{2}}$$

$$I(q^2) \equiv \int_{4m^2}^{\infty} d\kappa^2 \frac{1}{q^2 + \kappa^2 - i\varepsilon} \frac{1}{8\pi^2} [1 - \frac{4m^2}{\kappa^2}]^{\frac{1}{2}}$$

$$J^A(q^2) \equiv - \int_{4m^2}^{\infty} d\kappa^2 \frac{1}{q^2 + \kappa^2 - i\varepsilon} \frac{1}{12\pi^2} [1 - \frac{4m^2}{\kappa^2}]^{\frac{1}{2}} (\kappa^2 - 4m^2)$$

$$+ \int_{4m^2}^{\infty} d\kappa^2 \frac{1}{12\pi^2} [1 - \frac{4m^2}{\kappa^2}]^{\frac{1}{2}} (1 - \frac{2m^2}{\kappa^2})$$

$$J^V(q^2) \equiv \int_{4m^2}^{\infty} d\kappa^2 \frac{1}{q^2 + \kappa^2 - i\varepsilon} \frac{1}{12\pi^2} (1 - \frac{4m^2}{\kappa^2})^{\frac{1}{2}} (1 + \frac{2m^2}{\kappa^2})$$

$$J^S(q^2) \equiv \int_{4m^2}^{\infty} d\kappa^2 \frac{\kappa^2}{q^2+\kappa^2-i\varepsilon} \frac{1}{12\pi^2} (1 - \frac{4m^2}{\kappa^2}) \quad (1+\frac{2m^2}{\kappa^2})$$

$$-i(2\pi)^{-4} \int d^4p \; Tr[\gamma_5 S_c(p+\tfrac{1}{2}q)\gamma_5 S_c(p-\tfrac{1}{2}q)] = J^P(q^2)$$

$$-i(2\pi)^{-4} \int d^4p \; Tr[\gamma_5 S_c(p+\tfrac{1}{2}q) i\gamma_\mu\gamma_5 S_c(p-\tfrac{1}{2}q)]$$

$$= i(2\pi)^{-4} \int d^4p \; Tr[i\gamma_\mu\gamma_5 S_c(p+\tfrac{1}{2}q)\gamma_5 S_c(p-\tfrac{1}{2}q)]$$

$$= 2\, m\, q_\mu \; I(q^2)$$

$$-i(2\pi)^{-4} \int d^4p \; Tr[i\gamma_\mu\gamma_5 S_c(p+\tfrac{1}{2}q) i\gamma_\nu\gamma_5 S_c(p-\tfrac{1}{2}q)]$$

$$= J^A(q^2)[\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}] + 4m^2 I(q^2) \frac{q_\mu q_\nu}{q^2}$$

$$-i(2\pi)^{-4} \int d^4p \; Tr[\gamma_\mu S_c(p+\tfrac{1}{2}q)\gamma_\nu S_c(p-\tfrac{1}{2}q)]$$

$$= -i(2\pi)^{-4} \; 4\int d^4p \; \frac{\delta_{\mu\nu}(p^2-\frac{1}{4}q^2+m^2) - 2p_\mu p_\nu + \frac{1}{2}q_\mu q_\nu}{[(p+\frac{1}{2}q)^2+m^2-i\varepsilon][(p-\frac{1}{2}q)^2+m^2-i\varepsilon]}$$

$$= J^V(q^2)(q_\mu q_\nu - \delta_{\mu\nu}q^2) + J^S(q^2)\delta_{\mu\nu}$$

$$-i(2\pi)^{-4} \int d^4p \; Tr[S_c(p+\tfrac{1}{2}q) S_c(p-\tfrac{1}{2}q)]$$

$$= -i(2\pi)^{-4} 4\int d^4p \; \frac{-(p^2-\frac{1}{4}q^2)+m^2}{[(p+\frac{1}{2}q)^2+m^2-i\varepsilon][(p-\frac{1}{2}q)^2+m^2-i\varepsilon]}$$

$$= \frac{1}{2} J^S(0) - J^P(q^2) + 4m^2 I(q^2)$$

APPENDIX D

EVALUATION OF THE MASS CORRECTION FOR THE MODIFIED YUKAWA INTERACTION USING THE BETHE-SALPETER APPROACH

In this appendix we will calculate the mass correction, $\Delta\mu^2$, for the modified Yukawa interaction using the Bethe-Salpeter approach [Salpeter and Bethe, 1951; Gell-Mann and Low, 1951] rather than the propagator method we used in Chapter 4.

The Lagrangian (4.67) leads to the Euler-Lagrange equations:

$$
\begin{aligned}
(\gamma.\partial+m)\psi(x) &= 2ig\gamma_5\psi(x)(i\bar{\psi}(x)\gamma_5\psi(x)) \\
&\quad + \delta m\psi(x) + if\gamma_5\psi(x)\phi(x) \;, \\
&\equiv \eta(x) \;; \qquad\qquad (D.1)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\psi}(-\gamma\overleftarrow{\partial}+m) &= 2ig(i\bar{\psi}(x)\gamma_5\psi(x))\bar{\psi}(x)\gamma_5 \\
&\quad + \delta m\bar{\psi}(x) + if\bar{\psi}(x)\gamma_5\phi(x) \;, \\
&\equiv \bar{\eta}(x) \;; \qquad\qquad (D.2)
\end{aligned}
$$

$$
\begin{aligned}
(\Box - \mu^2)\phi(x) &= -if\bar{\psi}(x)\gamma_5\psi(x) - \Delta\mu^2\phi(x) \;, \\
&\equiv j(x) \;. \qquad\qquad (D.3)
\end{aligned}
$$

The Bethe-Salpeter (B-S) amplitude $\chi_q^{\alpha\beta}(x,y)$ is defined as

$$\chi_q^{\alpha\beta}(x,y) = <0|T(\psi_\alpha(x),\bar{\psi}_\beta(y))|q> \tag{D.4}$$

where $|q>$ is an eigenstate of the total energy-momentum operator, P_μ, i.e.

$$P_\mu|q> = q_\mu|q> \quad . \tag{D.5}$$

Making use of the equations of motion (D.1)-(D.3) we can derive the relationship

$$(\gamma.\partial_{(x)} + m)_{\alpha\alpha'} \chi_q^{\alpha'\beta'}(x,y)(-\gamma\overleftarrow{\partial}_{(y)} + m)_{\beta'\beta}$$

$$= -i\delta(x_o-y_o)(\gamma_4)^{\alpha\alpha'}<0|\{\eta_{\alpha'}(x),\bar{\psi}_\beta(y)\}|q>$$

$$+ <0|T(\eta_\alpha(x),\bar{\eta}_\beta(y))|q> \quad . \tag{D.6}$$

In the pair-approximation the last term can be ignored, then use of the equal-time commutation relations reduces (D.6) to

$$(\gamma\partial_{(x)} + m)_{\alpha\alpha'} \chi_q^{\alpha'\beta'}(x,y)(-\gamma\partial_{(y)} + m)_{\beta'\beta}$$

$$= -2ig\ \delta^4(x-y)(\gamma_5)^{\alpha\beta}\ \mathrm{Tr}[\gamma_5\chi_q(x,x)]$$

$$-if\ \delta^4(x-y)(i\gamma_5)_{\alpha\beta}\ <0|\phi(x)|q> \quad . \tag{D.7}$$

Integrating this equation, we obtain

$$\chi_q(x,y) = \chi_q^{(o)}(x,y)$$

$$- 2ig\int d^4u\ S_c(x-u)\gamma_5 S_c(u-y)\,\mathrm{Tr}[\gamma_5\chi_q(x,x)]$$

$$- if\int d^4u\ S_c(x-u)i\gamma_5 S_c(u-y)\langle 0|\phi|q\rangle \quad . \qquad \text{(D.8)}$$

The quantity $\chi_q^{(o)}(x,y)$ is the solution of the homogeneous equation

$$(\gamma\partial_{(x)} + m)\chi_q^{(o)}(x,y)(-\gamma\overleftarrow{\partial}_{(y)} + m) = 0 \quad . \qquad \text{(D.9)}$$

The first term on the right hand side of (D.8) represents the free-fermion contribution* and vanishes if $|q\rangle$ represents a bound state.

It is more convenient, at this stage, to factorize out the "centre of mass" dependency in the same way as we would for the Schrodinger wave function by making the change of variable

$$X = \frac{1}{2}(x + y) \qquad \text{(D.10)}$$

$$Z = (x - y) \quad . \qquad \text{(D.11)}$$

Expressed in terms of X and Z the B-S amplitude is

$$\chi_q^{\alpha\beta}(x,y) = (2\pi)^{-3/2}\, e^{iqx}\, \chi_q^{\alpha\beta}(z;q) \quad . \qquad \text{(D.12)}$$

---

* See, for example, D. Lurié, 1968, Chapter 9.

Equation (D.8) can then be rewritten as:

$$\chi_q(z) = \chi_q^{(o)}(z) + 2g\, \mathrm{Tr}[\gamma_5 \chi_q(0)]\, Q(z;q)$$

$$+ if\, \phi_q\, Q(z;q) \quad . \qquad \text{(D.13)}$$

Here we have made the change of notation

$$<0|\phi(x)|q> = (2\pi)^{-3/2}\, e^{iqx}\, \phi_q \qquad \text{(D.14)}$$

and

$$Q(z;q) = -i\int d^4y\, S_c(\tfrac{z}{2} - y)\gamma_5 S_c(\tfrac{z}{2} + y)\, e^{iqy} \quad . \qquad \text{(D.15)}$$

If we multiply (D.13) by γ_5 and take the trace of both sides we obtain:

$$\mathrm{Tr}[\gamma_5 \chi_q(0)] = \mathrm{Tr}[\gamma_5 \chi_q^o(0)] + 2g\, \mathrm{Tr}[\gamma_5 \chi_q(0)]\, J_p(q^2)$$

$$+ if\, \phi_q\, J_p(q^2) \quad . \qquad \text{(D.16)}$$

We have set $z = 0$ in (D.16) and introduced the pseudo-scalar-pseudoscalar closed loop integral $J_p(q^2)$.

Equation (D.16) can be further simplified by the introduction of $C(q)$,

$$C(q) \equiv \mathrm{Tr}(i\gamma_5 \chi_q(0)) \qquad \text{(D.17)}$$

whence,

$$C(q) = C^o(q) + 2g\, J_p(q^2) C(q) = f\, \phi_q\, J_p(q^2)$$

$$= C^o(q) + J_p(q^2)[2g\, C(q) - f\, \phi_q] \quad . \qquad \text{(D.18)}$$

If we only consider those states $|q\rangle$ that correspond to a bound state we can put $C^{o}(q)$ equal to zero:

$$[1 - 2g\, J_p(q^2)]C(q) = -f\, J_p(q^2)\, \phi_q \quad . \tag{D.19}$$

On the other hand, (D.3) say that

$$(q^2 + \mu^2)\phi_q = \Delta\mu^2\, \phi_q - f\, C(q) \quad , \tag{D.20}$$

so at $q^2 + \mu^2 = 0$

$$[1 - 2g\, J_p(-\mu^2)]C(q) = -f\, J_p(-\mu^2)\phi_q \quad , \tag{D.21}$$

$$-\Delta\mu^2\, \phi_q = -f\, C(q) \quad . \tag{D.22}$$

The condition for a solution to exist for these simultaneous equations is:

$$\begin{vmatrix} 1 - 2g\, J_p(-\mu^2) & f\, J_p(-\mu^2) \\ -f & \Delta\mu^2 \end{vmatrix} = 0 \tag{D.23}$$

or

$$\Delta\mu^2 = \frac{-f^2\, J_p(-\mu^2)}{[1 - 2g\, J_p(-\mu^2)]} \quad . \tag{D.24}$$

This result is in agreement with the result we obtained in Chapter 4.

If we substitute (D.24) into (D.20) and define $b(q^2)$ as we did in Chapter 4, i.e.

$$J_p(q^2) - J_p(-\mu^2) = (q^2 + \mu^2)b(q^2) \ , \qquad \text{(D.25)}$$

we find that the equation for the bound state mass is

$$(1 - 2g\ J_p(-\mu^2))(1 - 2g\ J_p(-\mu_b^2)) - f^2 b(-\mu_b^2) = 0 \ . \qquad \text{(D.26)}$$

This, again, is the same as the result we obtained in Chapter 4 using the propagator method.

REFERENCES

Anderson P.W., 1958, Physical Review 112, 1900.

Anderson P.W., 1963, Physical Review 130, 433.

Aurilia A., Y. Takahashi and H. Umezawa, 1971, "The spontaneous breakdown of symmetry and gauge invariance in relativistic field theory", University of Alberta Preprint.

Aurilia A., N.J. Papastamatiou, Y. Takahashi and H. Umezawa, 1972, "The spontaneous violation of the dilatation invariance in relativistic field theories", University of Alberta Preprint.

Chang P. and F. Gursey, 1969, Nuovo Cimento 63, 617.

Coombes C.A., 1968, Canadian J. of Physics 46, 929.

Ezawa H., 1963, Anns. of Physics 24, 46.

Fermi E. and C.N. Yang, 1949, Phys. Rev. 76, 1739.

Ferrari R. and L.E. Picasso, 1971, "Spontaneous breadown in Quantum Electrodynamics", Istituto di Fisica dell'Università di Pisa Preprint.

Fujii Y., 1959, Prog. Theoretical Phys. (Kyoto) 21, 232.

Gell-Mann M. and F.E. Low, 1951, Phys. Rev. 84, 350.

Goldstone J., 1961, Nuovo Cimento 19, 154.

Goldstone J., A. Salam and S. Weinberg, 1962, Phys. Rev. 127, 965.

Higgs P.W., 1964(a), Phys. Letters 12, 132.

Higgs P.W., 1964(b), Phys. Letters 13, 508.

Higgs P.W., 1966, Phys. Rev. 145, 1156.

Iwasaki Y., 1968, Phys. Rev. 173, 1608.

Jenkins J.D., 1972, "The Electromagnetic interaction of a massive spin-one particle", University of Durham, England, Preprint.

Jin Y.S. and S.W. MacDowell, 1965, Phys. Rev. 137, B688.

Källèn G., 1950, Arkiv. Fysik 2, 371.

Kibble T.W., 1967, Phys. Rev. 155, 1554.

Kyriakopoulos E., 1969, Phys. Rev. 183, 1318.

Lee T.D., 1954, Phys. Rev. 95, 1329.

Lee T.D. and C.N. Yang, 1955, Phys. Rev. 98, 1501.

Lurié D., 1968, "Particles and Fields", Interscience, London.

Lurié D., A.J. Macfarlane and Y. Takahashi, 1965, Phys. Rev. 140, B1091.

Macfarlane A.J. and W. Tait, 1971, "Tensor formulation of Spin-One and Spin-Two", University of Cambridge Preprint.

Nambu Y. and G. Jona-Lasinio, 1961, Phys. Rev. 122, 345.

Palmer, R.F. and Y. Takahashi, 1970, Phys. Rev. D2, 3086.

Roman P., 1964, "Theory of Elementary Particles", North-Holland, Amsterdam.

Roman P., 1969, "Introduction to Quantum Field Theory", John Wiley, New York.

Sakurai, J.J., 1960, Anns. of Phys. 11, 1.

Sakurai, J.J., 1969, "Currents and Mesons", Chicago U. Press, Chicago.

Salpeter E.E. and H.A. Bethe, 1951, Phys. Rev. 84, 1232.

Schweber S.S., 1961, "An Introduction to Relativistic Quantum Field Theory", Harper Row, New York.

Schwinger J., 1956, Phys. Rev. 104, 1164.

Sen R.N. and H. Umezawa, 1967, Nuovo Cimento 50, 53.

Shamaly A. and A.Z. Capri, 1971, "Propagation of interacting fields", University of Alberta Preprint.

Stueckelberg E.C.G., 1938, Helv. Phys. Acta 11, 225.

Takahashi Y., 1969, "An Introduction to Field Quantization", Pergamon Press, London.

Takahashi Y. and R.F. Palmer, 1970, Phys. Rev. D1, 2974.

Takahashi Y. and H. Umezawa, 1953, Prog. of Theoretical Phys. 9, 14.

Takahashi Y. and H. Umezawa, 1964, Nucl. Phys. 51, 193.

Umezawa H., 1956, "Quantum Field Theory", North-Holland, Amsterdam.

Umezawa H., 1965, Nuovo Cimento 40, 450.

Yang C.N. and D. Feldman, 1950, Phys. Rev. 79, 972.

Yang C.N. and R.L. Mills, 1954, Phys. Rev. 96, 191.